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REPORT SERIES

Reducibility of Domain Representations and Cantor–Weihrauch Domain Representations

by

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Report # CSR 15-2006

Computer Science
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Reducibility of Domain Representations and Cantor–Weihrauch Domain Representations

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September 27, 2006

Abstract

A notion of (continuous) reducibility of representations of topological spaces is introduced and basic properties of this notion are studied for domain representations.

A representation reduces to another if its representing map factors through the other representation. Reductions form a pre-order on representations. A spectrum is a class of representations divided by the equivalence relation induced by reductions. Representations belonging to the top element of a spectrum are said to be universal and these representations are the ones most closely capturing the structure of the represented space. Notion of admissibility considered both for domains and within Weihrauch's TTE are shown to be universality concepts in the appropriate spectra.

To illustrate the framework, some domain representations of real numbers are considered and it is shown that the usual interval domain representation, which is universal among dense representations, does not reduce to a binary expansion domain representation. However, a substructure of the interval domain more suitable for efficient computation of operations is on the other hand shown to be equivalent to the usual interval domain with respect to reducibility.

1 Introduction

A standard method of computing on a set X of data is to make a representation R of the data and to compute on R. Such methods have been called concrete computability theories [30, 31]. The question arises immediately; to what extent does computability on X depend on the *choice* of R? For any concrete computability there is the problem of clarifying the applicability of representations for different tasks.

One approach to resolve the above problem is to look at the intrinsic properties of representations as was done in, e.g., [6]. As an alternative approach, we present here a framework for relating representations to each other, thereby studying there relative merits. This is done by studying *reductions* (or *translations*) between representations. This is parallel to the use of reductions for *numberings* (used in Computable Algebra). Reductions between numberings (when a numbering factors through another) is one of the basic tools in studying numberings [15, 16, 17, 20, 27, 28]. We generalise reducibility to a very general class of representations of topological spaces and study basic properties of reducibility, in particular for domain representations.

Our aim is to study computability on uncountable structures (usually topological spaces). A simple numbering is not possible of an uncountable structure. We therefore have to rely on computations on some numbered set of approximations. For example, real number computations can be performed using the countable set of rational intervals as approximations. A general method of giving computability theory to a large class of topological spaces is to use *domain representations*.

Representations of topological spaces by domains or embeddings of topological spaces into domains have been studied by several people [2, 3, 4, 10, 11, 12, 13, 14, 18, 21, 24, 26, 27, 32, 35]. Domain representations are also closely related to Type-2 Theory of Effectivity (TTE) [29, 33, 34] introduced by Weihrauch.

Any T_0 space can be given domain representations [6]. Some of these have nice properties such as density and an embedding property. These properties facilitate lifting of functions to the domain representations, thereby opening up for a study of topological algebras.

Our reducibility notion introduces a pre-order on domain representations and thereby an equivalence relation. A *spectrum* is a class of representations divided by the equivalence relation. We give examples showing that the structure of the spectrum of all representations is in general non-trivial.

Some intrinsic properties of domain representations interact with our theory of reductions, so, for example, the representations that have the embedding property are known as *retract* representations and these are invariant under reductions.

The importance of density in domain representations has an information theoretic explanation in that non-dense representations contain non-consistent information or “garbage”. When restricting our attention to dense representations, there is a top element in the spectrum of all dense domain representations, namely the equivalence class of dense retract representations.

If there exists a top element in a spectrum then we call a representation belonging to it *universal* as all other representations in that spectrum will reduce to it. We show that notions of admissibility as studied by, for example, Schröder [23] and Hamrin [18], are in fact notions of universality in the appropriate spectrum. Universal representations capture the structure of the represented space closest among the class of representations considered.

To illustrate the framework, we conclude by studying some representations of real numbers. The usual interval domain representation of the reals is known to be universal among countably based dense representations (ω-admissible). We show that a particular substructure of the interval domain, where operations on exact reals can be more efficiently computed, is continuously equivalent to the interval domain (although it is a bifinite domain rather than a Scott–Eršov domain), and hence that it can be used interchangeably. Finally, we show that a representation corresponding to binary expansions is not a universal representation of the reals. This is highlighted by the example showing that addition is not a computable operation on the binary expansions of reals.

I thank John V. Tucker, Viggo Stoltenberg-Hansen and Fredrik Dahlgren for many invaluable discussions on this paper.

2 Preliminaries

2.1 Domains

We will briefly give some background to domain theory. For a fuller background on domains we refer to [1, 25].

Let $D = (D, \sqsubseteq)$ be a partially ordered set. A subset $A \subseteq D$ is an *upper set* if $x \in A$ and $x \sqsubseteq y$ implies $y \in A$. Let $\uparrow A = \{y \in D : \exists x \in A(x \sqsubseteq y)\}$. We will abbreviate $\uparrow\{x\}$ by $\uparrow x$. A subset $A \subseteq D$ is *directed* if $A \neq \emptyset$ and whenever $x, y \in A$ then there is $z \in A$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. The supremum, or least upper bound, of A (if it exists) is denoted by $\bigsqcup A$.

A *complete partial order*, abbreviated *cpo*, is a partial order, $D = (D; \sqsubseteq, \bot)$, such that $\bot$ is the least element in D and any directed set $A \subseteq D$ has a supremum, $\bigsqcup A$. This is also known as a pointed *dcpo* in the literature.

Let D be a cpo. An element x is *way below* y, denoted $x \ll y$, if for each directed $A \subseteq D$,

$$y \sqsubseteq \bigsqcup A \implies (\exists z \in A)(x \sqsubseteq z)\,.$$

An element $a \in D$ is *compact* if $a \ll a$. The set of compact elements of D is denoted by D_c.

A subset B of a cpo D is a *basis* for D if for every $x \in D$ the set $\downarrow\!\!\downarrow x \cap B$ is directed and has supremum x. A cpo D is a *continuous cpo* if it has a basis, and an *algebraic cpo* if D_c is a basis for D.

A cpo D is *consistently complete* if $\bigsqcup A$ exists in D whenever $A \subseteq D$ is a consistent set, i.e., has an upper bound.

Definition 2.1. A *Scott–Eršov domain*, or simply *domain*, is a consistently complete algebraic cpo.

The topology normally used on domains is called the Scott topology. Let D be an algebraic cpo. A subset U of D is open if

1. U is an upper set, and
2. $x \in U$ implies that there exists $a \in \mathrm{approx}(x)$ such that $a \in U$.

An easy observation is that the Scott topology on a domain is T_0. However the Scott topology fails to be T_1 on all domains except the trivial domain consisting of a single element.

The sets $\uparrow a$, for $a \in D_{\mathrm{c}}$, constitute a base for the Scott topology on a domain D.

Let D and E be domains. A function $f : D \to E$ is Scott continuous if f is monotone and

$$f(\bigsqcup A) = \bigsqcup f[A],$$

for any directed $A \subseteq D$. The notion of Scott continuity coincides with the notion of continuity induced from the Scott topology on the domains.

Any continuous function between domains is determined by its values on the compact elements.

Let D be an algebraic cpo, E a cpo, and let $f : D_{\mathrm{c}} \to E$ be a monotone function. Then there exists a unique continuous function $g : D \to E$ of f such that $f = g|_{D_{\mathrm{c}}}$.

Domains are often constructed as the completion of some underlying structure. We present here the type of structure from which Scott–Eršov domains are constructed.

The compact elements D_{c} of a Scott–Eršov domain D form a conditional upper semilattice with least element, abbreviated *cusl*. That is, a cusl is a partially ordered set where a least upper bound exists for every pair of elements that have an upper bound.

An *ideal* is a directed lower set. The ideal completion over a cusl P is the set of all ideals over P, denoted $\mathrm{Idl}(P)$. When ordered by set inclusion the ideal completion of a cusl forms a Scott–Eršov domain. For a in a cusl P, $\downarrow a$ is an ideal, the *principal ideal* generated by a. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals $\downarrow a$, for $a \in P$.

The representation theorem for Scott–Eršov domains tells us that any Scott–Eršov domain is the ideal completion of a cusl.

Theorem 2.2. *Let D be a Scott–Eršov domain. Then $\mathrm{Idl}(D_{\mathrm{c}}) \cong D$.*

We clearly have the following equivalence, for $I \in \mathrm{Idl}(P)$

$$\downarrow a \subseteq I \iff a \in I.$$

Thus the sets $B_a = \{I \in \mathrm{Idl}(P) : a \in I\}$ for $a \in P$ form a base for the Scott topology on $\mathrm{Idl}(P)$.

2.2 Representations

We give some background on representations of spaces. We give a more general setting than the domain representations considered in [6], but we still aim for representations using some type of domain. The terminology is adjusted to cope with a more general framework.

Definition 2.3. (i) A *weak representation* of a topological space X is a triple $(D, D^{\mathrm{R}}, \rho)$, where D is a topological space, $D^{\mathrm{R}} \subseteq D$ with the subspace topology, and $\rho : D^{\mathrm{R}} \to X$ is continuous and onto.

(ii) A *quotient representation* is a weak representation where ρ is a quotient map.

The word representation will be used without qualification to mean a weak representation.

In [6] this notion of representation is studied where D is required to be a domain. We will always have domain representations in mind, but define the notion as general as possible.

When needed, we write *continuous cpo representation*, *domain representation*, etc., to specify the kind of space that D is. We will primarily focus on Scott–Eršov domains and algebraic cpos, since by Proposition 2.8 any continuous cpo representation can be used to construct an algebraic cpo representation without losing any property considered herein.

The introduced notion of representations above covers all *naming systems* used in TTE, i.e., both *notations* and *representations*. In fact, all TTE naming systems can be constructed as Scott–Eršov domain representations using simple and specific domains.

The set D^{R} above will be called the set of *representing elements*. For a domain-like structure D the set D^{R} is also known as a *totality* on D. If D is a domain then the ordering of the domain can be interpreted as an information ordering. With this interpretation the domain contains both proper approximations and total or complete representations of elements of X, the latter constituting the set D^{R}. Intuitively, D^{R} consists of those domain elements that contain sufficient information to completely determine an element in X via ρ.

Beyond the type of space D used in a representation, we make use of the following important characteristics of representations.

Definition 2.4. (i) A representation $(D, D^{\mathrm{R}}, \rho)$ is *dense* if D^{R} is dense in D.

(ii) A *retract representation* of X is a quadruple $(D, D^{\mathrm{R}}, \rho, \eta)$ where $(D, D^{\mathrm{R}}, \rho)$ is a representation, and $\eta : X \to D^{\mathrm{R}}$ is a continuous function such that $\rho\eta = \mathrm{id}_X$.

For a retract representation $(D, D^{\mathrm{R}}, \rho, \eta)$ we have that ρ is a quotient, and that $\eta\rho$ is a retraction on D^{R}. In fact, X will be homeomorphic to the retract of D^{R}. In a retract representation a canonical representative can be found continuously from any representation of an element of X.

Definition 2.5. Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of X and Y respectively. A continuous function $f : X \to Y$ is *represented* by a continuous function $\bar{f} : D \to E$ if $\rho_E \bar{f}(x) = f\rho_D(x)$, for all $x \in D^{\mathrm{R}}$ (in particular, $\bar{f}[D^{\mathrm{R}}] \subseteq$

E^{R}).

$$
\begin{array}{ccc}
D & \xrightarrow{\bar{f}} & E \\
\iota \uparrow & & \uparrow \iota \\
D^{\mathrm{R}} & \xrightarrow{\bar{f}} & E^{\mathrm{R}} \\
\rho_D \downarrow & & \downarrow \rho_E \\
X & \xrightarrow{f} & Y
\end{array}
$$

The functions between the subsets of representing elements are restrictions of functions. To avoid clumsy explicit restriction notation, as in $\bar{f}|_{D^{\mathrm{R}}} : D^{\mathrm{R}} \to E^{\mathrm{R}}$, we write $\bar{f} : D^{\mathrm{R}} \to E^{\mathrm{R}}$ and trust the reader to understand this as the restriction to the indicated domain of the function.

Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of X and Y respectively, and let $\bar{f} : D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$. If $\bar{f}$ respects the equivalence relations induced by ρ_D and ρ_E, then $\bar{f}$ represents a well-defined function $f : X \to Y$. Furthermore, if ρ_D is a quotient map, then f is continuous, since then the topology of X is fine enough.

For a topological space X we define the following classes of representations of X.

Definition 2.6. For a topological space X let $\mathbf{Rep}(X)$ denote the class of all representations of X, and let $\mathbf{DRep}(X)$ denote the class of all domain representations $(D, D^{\mathrm{R}}, \rho)$of X where the totality D^{R} does not contain $\perp$.

The reason to disallow domain representation where $\perp$ is a representing element is admittedly technical but is needed for the proof of Lemma 3.19. Note that $\perp$ cannot belong to the totality if every point of the space X has a non-trivial neighbourhood base, which is the case for all T_1 spaces with at least two points. Also note that if $(D, D^{\mathrm{R}}, \rho)$ is a domain representation of X, then so is the lifting $(D_\perp, D^{\mathrm{R}}, \rho)$, and in the latter $\perp_{D_\perp}$ does not belong to the totality.

We also consider subclasses of these classes of representations. In particular, subclasses of dense representations and subclasses of representations of bounded size.

Definition 2.7. Let $\mathbf{R}(X)$ be a class of representations of X.

1. $\mathbf{R}^{\mathrm{D}}(X)$ is the subclass of $\mathbf{R}(X)$ containing all dense representations.

2. Fix κ to be an infinite cardinal, then $\mathbf{R}_\kappa(X)$ is the subclass of $\mathbf{R}(X)$ containing all representations where the topological space D has a base with cardinality bounded by κ.

For domain representations the above definition of representations of bounded size can be rephrased as a bound on the set of compact elements. For example, $\mathbf{DRep}_\kappa(X)$ is the class of all κ-based domain representations, i.e., representations where the cardinality of the compact elements of the domain is bounded by κ.

The most interesting class of domain representations with bounded cardinality is, of course, the class of countably based domain representations, i.e., $\mathbf{DRep}_\omega(X)$, since these are the ones to which effectivity can be applied.

We repeat some of the results from [6]. The following proposition sums up the results in Section 4 of [6] and shows why we may restrict our attention to representations from algebraic cpos.

Proposition 2.8. *Let $(D, D^{\mathrm{R}}, \rho_E)$ be a continuous cpo representation of X. Then there is a canonical algebraic cpo representation $(E, E^{\mathrm{R}}, \rho_E)$ retaining the properties of quotient, retract, and density if present in the original representation.*

For full proofs of the following theorems, see Theorems 5.4, 5.6, and 9.3 of [6] respectively.

Theorem 2.9. *Any T_0 space has a dense retract domain representation.*

Proof. We give here briefly the construction. Let B be a base of the topology for a T_0 space X. Let B' be the non-empty sets of B together with the set X. Ordering B' by reverse inclusion gives a cusl, and the ideal completion of this cusl is a domain D. The function $\eta : X \to D$ given by $\eta(x) = \{U \in B' : x \in U\}$ is an embedding.

It remains to be shown that $(D, \eta[X], \eta^{-1}, \eta)$ is a dense retract domain representation of X. □

Remark 2.10. By allowing B' in the above construction to contain the empty set the domain representation will have a compact top element but will otherwise be identical. Constructing the domain representation of X as above we get a non-dense (since the top element is not in $\eta[X]$) retract domain representation of the space X.

Theorem 2.11. *A space with a retract cpo representation is a T_0 space.*

Theorem 2.12. *Let $(D, D^{\mathrm{R}}, \rho_D)$ be a dense domain representation of X, and let $(E, E^{\mathrm{R}}, \rho_E, \eta_E)$ be a retract domain representation of Y. Then every continuous function $f : X \to Y$ is represented by some continuous function $\bar{f} : D \to E$.*

Proof. The construction of $\bar{f}$ is done in two steps. First, let $f' = \eta_E f \rho_D$. The function $f' : D^{\mathrm{R}} \to E$ is a continuous function representing f.

Secondly, the function f' is extended to a function $\bar{f}\colon D_c \to E$ by $\bar{f}(a) = \sqcap f'[\uparrow a \cap D^{\mathrm{R}}]$. The infimum is well-defined since $\uparrow a \cap D^{\mathrm{R}}$ is non-empty by density of D^{R}, and non-empty infima exist in consistently complete cpos. Clearly, $\bar{f}$ is monotone, and hence, it has a unique extension to D.

It remains to be shown that $\bar{f}$ is an extension of f', i.e., that $\bar{f}(d) = f'(d)$ for $d \in D^{\mathrm{R}}$. □

Remark 2.13. Density is needed in order to give a well-defined infimum in the second step of the construction of the lifting. The same purpose can also be achieved if E has a top element, since we may define the infimum of the empty set to be the top element of E. Hence, the above result could be stated for arbitrary D and a retract representation E with a top element.

Other lifting results where density of D is not required can be found in [7, 19, 22]

2.3 Partial Domain Functions

We will use the notion of continuous partial domain functions introduced by Dahlgren [9].

Definition 2.14. Let D and E be domains. A *continuous partial function* from D to E is a pair (S, f) where $S \subseteq D$ is a non-empty closed subset of D, and f is a strict continuous function from S to E.

Note that a total domain function is not necessarily a partial domain function. In fact, only strict total functions are partial functions. The strictness of partial functions is required to make composition always defined. (Allowing totally undefined partial functions would solve composition but entails that the category of domains with partial functions does not have a terminal object. The empty domain might be added as the terminal object of the category but that would make the definition of products hard.)

The closed set S is downwards closed and must therefore contain $\bot$. The closed set that is of principal interest to us is the closure of the totality of a domain representation, that is, the closed subset S containing the totality of the domain and such that the totality is dense in S.

Recall that a common interpretation of the ordering relation of a domain representation is that it corresponds to information. High up in the domain means much information about an object of the space. So a non-dense representation can be viewed as a representation that contains non-consistent information or "garbage". In practise it is sometimes desirable to cut away this non-consistent information, i.e., to restrict to a substructure (see, for example, Lemma 2.28 in [5]).

This restriction can be done in general for domain representations by taking the ideal completion over all approximations of representing elements. Formally, for a domain representation $(D, D^{\mathrm{R}}, \rho)$ of X let

$$D_{\mathrm{c}}^{\mathrm{D}} = \{a \in D_{\mathrm{c}} : \uparrow a \cap D^{\mathrm{R}} \neq \emptyset\}$$

and

$$D^{\mathrm{D}} = \{\bigsqcup A : A \subseteq D_{\mathrm{c}}^{\mathrm{D}} \text{ is directed}\}\,.$$

Clearly, D^{D} is a closed subset of D. Ordering D^{D} by the order of D make D^{D} into a domain. The set D^{D} contains D^{R}. Thus, $(D^{\mathrm{D}}, D^{\mathrm{R}}, \rho)$ is a dense domain representation of X, which we will refer to as the *dense part* of D.

Proposition 2.15. *Let D be a domain representation in $\mathbf{DRep}(X)$, and E a retract domain representation of Y. Then any function $f : X \to Y$ is representable by a continuous partial domain function $\bar{f} : D \rightharpoonup E$, where $\bar{f}$ is defined on D^{D}.*

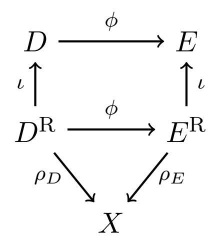
Proof. By Theorem 2.12 there exists a continuous functions $f' : D^{\mathrm{D}} \to E$. Construct a strict function $\bar{f}$ by

$$\bar{f}(c) = \begin{cases} \bot, & \text{if } c = \bot; \\ f'(c), & \text{otherwise}. \end{cases}$$

As $\bot \notin D^{\mathrm{R}}$ we have that $\bar{f}(x) = f'(x)$ for all $x \in D^{\mathrm{R}}$. □

3 Reducibility

In order to study representations and their applicability to various tasks we give here a notion of reduction between representations of a fixed space. For representations D and E of a space X we have that D reduces to E if the representation function of D factors through the representation function of E, i.e., if there is a function $\phi : D \to E$ such that the following diagram commutes.



Another way of interpreting the above diagram is that the identity on X is representable, see Lemma 3.4.

We now give our formal definitions of reducibility between representations.

Definition 3.1. Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of a topological space X, and let $\phi : D \to E$ satisfy $\phi[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and $\rho_D(d) = \rho_E \phi(d)$.

(i) ϕ is a *continuous reduction* of D to E, written $D \leq_{\mathrm{c}} E$, if ϕ is continuous.

(ii) ϕ is a *continuous partial reduction* of D to E, written $D \leq_{\mathrm{cp}} E$, if ϕ is a partial continuous function such that $D^{\mathrm{R}} \subseteq \operatorname{dom} \phi$.

We have suffixed the relation with a c for *continuous* as we intend to consider effective reductions in the future.

Our definition of continuous partial reductions is not always unique as there may be more than one relevant notion of partiality depending on the kind of the spaces D and E. In particular, if D and E are domains we assume that the notion of partiality is the one considered above.

Definition 3.2. The representations $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ are *(continuously) equivalent,* $D \equiv_{\mathrm{c}} E$, if $D \leq_{\mathrm{c}} E$ and $E \leq_{\mathrm{c}} D$. Similarly for $\equiv_{\mathrm{cp}}$.

Lemma 3.3. *The relations $\leq_{\rm c}$ and $\leq_{\rm cp}$ are pre-orders. The relations $\equiv_{\rm c}$ and $\equiv_{\rm cp}$ are equivalence relations.*

Proof. The reduction relations are reflexive since the identity is a continuous function reducing a representation to itself. Transitivity follows by composition.

That reduction equivalences are equivalence relations follows from the definition by them in terms of the pre-orders. □

Lemma 3.4. *For representations D and E of X and for any of the above notions of reduction we have that D reduces to E if, and only if, the identity function on X is represented by some function of the appropriate type from D to E.*

Proof. Any reduction function represents the identity function on X, and the identity is continuous on any topological space X. Any function representing the identity on X is a reduction function. □

Since reductions introduce a pre-order, $\leq$, and an equivalence relation, $\equiv$, on a class of representations the natural objects to study are equivalence classes of representations. These will have a well-defined partial order induced by the pre-order. We call this structure the spectrum over a space.

Definition 3.5. A *spectrum* over a topological space X, written $\mathrm{Spec}(X, \mathcal{D}, \leq)$, is the quotient $\mathcal{D}/\!\equiv$ ordered by $\leq$, where $\mathcal{D}$ is a class of representations of the space X and $\leq$ is some pre-order on $\mathcal{D}$.

Spectrums are built in the same way as *degree structures* are built in Computability Theory.

Definition 3.6. Given a reduction relation $\leq$ we say that a domain representation D is *universal* in a class $\mathcal{D}$ of representations with respect to $\leq$ if every $E \in \mathcal{D}$ reduces to D, i.e., if $E \leq D$ for all $E \in \mathcal{D}$.

Thus, D is universal in a class of representations if D belongs to the largest equivalence class of the spectrum if such a class exists.

3.1 Category of representations

This section will give an alternative view of our investigation which the reader may skip if so inclined. However, we will refer to concepts herein occasionally.

Construct a category $\mathbf{Rep}$ of representations by letting the objects be tuples $(D, D^{\rm R}, \rho, X)$, where $(D, D^{\rm R}, \rho)$ is a representation of some space X. The arrows are functions between representations respecting the equivalence relation induced by the representing functions. Two arrows are equal if they represent the same function on the represented spaces.

The identity arrow on $(D, D^{\rm R}, \rho, X)$ is the identity on D. Composition of arrows is composition of functions. Thus, we have a category.

We may consider subcategories of the category $\mathbf{Rep}$, for example, domain representations $\mathbf{DRep}$, dense domain representations $\mathbf{DRep}^{\mathrm{D}}$, countably based domain representations $\mathbf{DRep}_{\omega}$, etc.

We may of course also consider the category of representations with arrows being continuous partial functions, e.g., domain representations with continuous partial domain functions as arrows, $\mathbf{pDRep}$.

The properties of reductions of representations for some topological space X can be studied as the subcategory $\mathbf{Rep}(X)$ of representations of X where the arrows are functions representing the identity on X, i.e., exactly the reductions. Likewise for domain representations, etc.

As examples of the categorical interpretation of our notions we give the following lemmas, where $\mathcal{D}$ is assumed to be a category of representations of a space X with arrows being reductions between representations.

As reductions represent the identity on X we have that any arrows between two representations are in fact equal. So if there is an arrow $f : D \to E$ in the category $\mathcal{D}$ it is necessarily unique.

First we show that a spectrum as introduced above is a category, where representations belonging to the same equivalence class in the spectrum will be isomorphic in the categorical sense.

Lemma 3.7. *Representations of $\mathcal{D}$ that are equivalent with respect to reductions are isomorphic.*

Proof. If D and E are equivalent, then there exist reductions in both directions, that is there are arrows $f : D \to E$ and $g : E \to D$ and as noted above these are necessarily unique. The composition $g \circ f : D \to D$ is an arrow and $g \circ f = 1_D$ by uniqueness of 1_D. Likewise, $f \circ g = 1_E$. □

Lemma 3.8. *A representation $(D, D^{\mathrm{R}}, \rho)$ in $\mathcal{D}$ is universal if, and only if, $(D, D^{\mathrm{R}}, \rho, X)$ is a terminal object in the category $\mathcal{D}$.*

Proof. For a universal representation $D \in \mathcal{D}$ we have a reduction from any other representation E to D by definition, i.e., an arrow from E to D which is again unique as noted above. Thus, there is a unique arrow from any object of the category to D showing that D is a terminal object.

Conversely, as arrows are reductions we have that any terminal object of the category $\mathcal{D}$ is necessarily universal in D. □

We will see in Section 3.3 that $\mathbf{DRep}(X)$ and $\mathbf{DRep}^{\mathrm{D}}(X)$ has terminal objects. In fact, there are also initial objects in these categories, namely the flat domain representation $(X_{\perp}, X, \mathrm{id})$. So, an initial representation represents the space simply as a discrete set, whereas a terminal representation represents as much of the structure of the space as possible within the particular class of representations.

3.2 Continuous Reductions

We give some basic results for continuous (total) reductions of domain representations, that is, let us study $\mathrm{Spec}(X, \mathbf{DRep}(X), \leq_{\mathrm{c}})$.

Theorem 3.9. *Let D be a dense domain representation of X, and E be a retract domain representation of X. Then D continuously reduces to E.*

Proof. By Theorem 2.12 the identity function can be lifted to a continuous domain function. By Lemma 3.4 D reduces to E. ☐

We note that all dense retract representations belong to the same equivalence class.

Corollary 3.10. *Dense retract domain representations are unique up to $\equiv_{\mathrm{c}}$.*

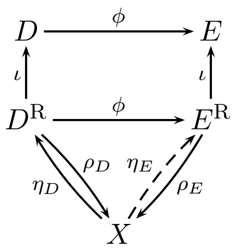
Proof. Immediate. ☐

The equivalence class of dense retract domain representations is central because any continuous function is representable if the spaces have dense retract domain representations by Theorem 2.12, and because dense retract domain representations exist for all T_0 spaces by Theorem 2.9. However, note that the equivalence class of dense retract representations also contains non-dense retract representations.

The following lemma shows that the property of retract is invariant under reductions.

Theorem 3.11. *Let $D \leq_{\mathrm{c}} E$ be representations of X. If D is a retract representation, then so is E.*

Proof. Let $(D, D^{\mathrm{R}}, \rho_D, \eta_D)$, and $(E, E^{\mathrm{R}}, \rho_E)$ be the representations. By reducibility there exists a continuous $\phi : D \to E$ such that $\rho_D = \rho_E\, \phi$.

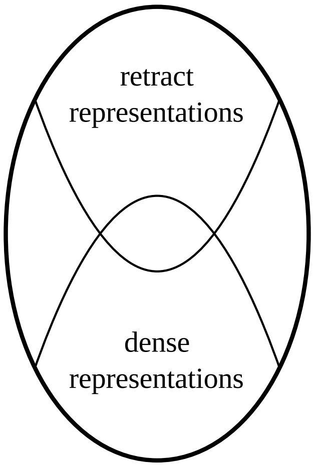


Let $\eta_E = \phi\, \eta_D$, then

$$\rho_E\, \eta_E = \rho_E\, \phi\, \eta_D = \rho_D\, \eta_D = \mathrm{id}_X$$

showing that $(E, E^{\mathrm{R}}, \rho_E, \eta_E)$ is a retract representation. ☐

Theorems 3.9 and 3.11 show that the spectrum $\mathrm{Spec}(X, \mathbf{DRep}(X), \leq_c)$ can roughly be drawn as follows, although equivalence classes of dense representations do contain non-dense representations.



The following examples show that the spectrum $\mathrm{Spec}(X, \mathbf{D}(X), \leq_c)$ in general is non-trivial. The first example shows that for non-discrete spaces there exist domain representations that are not retracts and hence strictly below the equivalence class containing dense retract representations.

Example 3.12. Let X be a non-discrete space. Clearly the flat domain $X_\perp$ gives a dense domain representation $(X_\perp, X, \mathrm{id})$ of X. The only possible embedding $\iota : X \to X_\perp$ is not continuous since there exists a subset $A \subseteq X$ that is not open in X whereas A is open in $X_\perp$.

The interested reader can (given a suitable choice principle) show that if the space X is discrete than any domain representation of X is a retract.

The following example shows that there exists non-dense retract representations that cannot be continuously reduced to any dense retract representation; and hence strictly above the equivalence class containing dense retract representations.

Example 3.13. Let X be a T_0 space containing at least two points separable by open sets. By Theorem 2.9 there exists a dense retract domain representation $(D, \eta[X], \rho, \eta)$ of X, and by Remark 2.10 there also exists a non-dense retract domain representation $(D_\top, \eta[X], \rho, \eta)$ with a compact top element.

The embedding $\iota : D \to D_\top$ is a continuous reduction of D to $D_\top$.

That X contains two points separable by open sets implies that the domain D contains non-consistent compact elements. Hence, a continuous and therefore monotone function cannot reduce $D_\top$ to D. Thus, $D <_c D_\top$.

The above two examples can be summarised by saying that unless the topology of the space X is extremely fine or extremely coarse there exist representations strictly below and strictly above, respectively, the equivalence class of dense retract representations.

The following example show that the equivalence class of any dense representations also contain non-dense representations.

Example 3.14. Let $(D, D^{\mathrm{R}}, \rho)$ be a dense representation of X. Construct a new domain representation by taking the disjoint sum of D and the trivial domain $\{\bot\}$ consisting of only the bottom element, i.e., $(D \oplus \{\bot\}, D^{\mathrm{R}}, \rho)$. Clearly, this representation is non-dense.

The embedding of D into the disjoint sum is a continuous reduction. Conversely, the projection of the disjoint sum onto D (mapping all of the right-hand domain to bottom in D) is a continuous reduction of the disjoint sum to D. Hence $D \equiv_{\mathrm{c}} D \oplus \{\bot\}$.

3.3 Universal and Retract Representations

Since the retract property is closed upwards in a spectrum it is a notion of being large in the spectrum, but clearly, so is also the notion of universality. We compare these two notions.

Theorem 3.15. *Consider the spectrum* $\mathrm{Spec}(X, \mathcal{D}, \leq_{\mathrm{c}})$*, where* $\mathcal{D}$ *is a class of domain representations. If there exists a retract representation in* $\mathcal{D}$ *then any universal representation is a retract representation.*

Proof. Let $D \in \mathcal{D}$ be a retract representation and let $E \in \mathcal{D}$ be a universal representation. By universality $D \leq_{\mathrm{c}} E$ so by Lemma 3.11 E is a retract representation. □

There exist universal representations in $\mathbf{DRep}(X)$ when X is a T_0 space. These representation have a compact top element that is not total. The top element could be interpreted as explicit inconsistent information.

Proposition 3.16. *Any* T_0 *space* X *has a universal domain representation* $(D_\top, D^{\mathrm{R}}, \rho)$ *in* $\mathbf{DRep}(X)$ *with respect to* $\leq_{\mathrm{c}}$.

Proof. Construct the domain representation $(D_\top, D^{\mathrm{R}}, \rho)$ as indicated in Remark 2.10. Reductions from any domain representation E of X can now be constructed as indicated in Remark 2.13. □

It is awkward to have a top element in a domain representation as very few functions have representations by continuous *total* domain functions. Restricting the representations to be dense, there still are universal representations and these include the dense retract representations.

Theorem 3.17. *Let* X *be a* T_0 *space. There exists a universal representation of* X *in* $\mathbf{DRep}^{\mathrm{D}}(X)$ *with respect to* $\leq_{\mathrm{c}}$*; and a representation is universal if, and only if, it is a retract.*

Proof. By Theorem 2.9 there exists a dense retract domain representation D of the space and by Theorem 3.9 any dense representation reduces to D, so D is universal, and by Corollary 3.10 any dense retract is universal.

The converse is Theorem 3.15. □

3.4 Partial Reductions

We now turn our attention to $\mathrm{Spec}(X, \mathbf{DRep}(X), \leq_{\mathrm{cp}})$. The first result is that any domain representation is continuously partially equivalent to its dense part.

Proposition 3.18. *A domain representation $(D, D^{\mathrm{R}}, \rho_D)$ of X is equivalent to its dense part $(D^{\mathrm{D}}, D^{\mathrm{R}}, \rho_D)$ with respect to $\equiv_{\mathrm{cp}}$.*

Proof. The partial map $\mathrm{id} : D \rightharpoonup D^{\mathrm{D}}$ defined on the closed set D^{D} is a partial domain functions reducing D to D^{D}. The embedding $\iota : D^{\mathrm{D}} \to D$ is a partial continuous reduction of D^{D} to D (that happens to be total). Hence $D \equiv_{\mathrm{cp}} D^{\mathrm{D}}$. □

The proposition implies that any element of the spectrum $\mathrm{Spec}(X, \mathbf{D}(X), \leq_{\mathrm{cp}})$ will contain a dense representation. This simplifies the structure of the spectrum as there will be fewer equivalence classes in general.

Lemma 3.19. *Let D and E be domain representations in $\mathbf{DRep}(X)$. Then $D \leq_{\mathrm{c}} E$ implies $D \leq_{\mathrm{cp}} E$.*

Proof. The bottom element $\perp$ does not belong to the totality of D. A total reduction function $f : D \to E$ can therefore be made into a continuous strict domain function (i.e., an everywhere defined continuous partial domain function) simply by changing its value at $\perp$ if necessary. □

Theorem 3.20. *Any retract domain representation D of X in $\mathbf{DRep}(X)$ is universal in $\mathbf{DRep}(X)$ with respect to $\leq_{\mathrm{cp}}$.*

Proof. Let E be an arbitrary domain representation of X. By Proposition 2.15 the identity function on X can be lifted to a continuous partial domain function $\phi : E \rightharpoonup D$ defined on E^{D}. By Lemma 3.4 $E \leq_{\mathrm{cp}} D$. □

We note that all retract representations belong to the same equivalence class.

Corollary 3.21. *Retract domain representations are unique up to $\equiv_{\mathrm{cp}}$.*

Proof. Immediate. □

Theorem 3.22. *Let X be a T_0 space. A representation D of X is universal in $\mathbf{pDRep}(X)$ if, and only if, it is a retract representation.*

Proof. By Theorem 3.20 a retract representation is universal. For the other direction assume that D is universal. By Theorem 2.9 there exists a dense retract representation E and by universality $E \leq_{\mathrm{cp}} D$. But since E is dense the reduction is in fact a (total) continuous reduction. Hence, by Lemma 3.11, D is a retract representation. □

Theorem 3.23. *Let X be a T_0 space.*

(i) A dense representation D is universal in $\mathbf{DRep}(X)^{\mathrm{D}}$ if, and only if, D is universal in $\mathbf{pDRep}(X)$.

(ii) If D is universal in $\mathbf{DRep}(X)$ then D^{D} is universal in $\mathbf{pDRep}(X)^{\mathrm{D}}$.

Proof. (i): By Theorem 3.17, the dense representation D is universal in $\mathbf{DRep}(X)$ if, and only if, D is a retract; and by Theorem 3.22, D is a retract if, and only if, D is universal in $\mathbf{pDRep}(X)$.

(ii): Again by Theorem 3.22, D is a retract, so D^{D} is a dense retract and hence universal in $\mathbf{pDRep}(X)^{\mathrm{D}}$. □

4 Cantor–Weihrauch Domain Representations

4.1 Cantor Domain Representations

Let Σ be an *alphabet*, i.e., a finite set of symbols containing at least two symbols. The Kleene star is an operator giving the set of all finite sequences over a set, so Σ^* is the space of all finite sequences over Σ. Let Σ^ω denote the set of all countably infinite sequences over Σ. Concatenation of sequences are denoted by juxtaposition.

Cantor space is the set Σ^ω with a base for the topology given by the sets

$$\{p\sigma : p \in \Sigma^*,\ \sigma \in \Sigma^\omega\}\,.$$

Note that the constructed Cantor space does not depend upon the choice of the alphabet Σ, i.e., even though the alphabets may differ, the Cantor spaces are homeomorphic.

We will now build a domain representation of Cantor space. The *Cantor domain*, denoted $\mathcal{C}$ is the set $\Sigma^* \cup \Sigma^\omega$ with the prefix ordering. It is easy to check that the Cantor domain is an algebraic cpo with Σ^* as its set of compact elements. Two elements of the Cantor domain are consistent only if one is a prefix of the other, so it is clearly consistently complete. There exists a numbering of Σ^* making the Cantor domain into an effective domain.

Cantor space is embedded densely into the Cantor domain as its non-compact elements, so we have the following result.

Proposition 4.1. *The Cantor domain is an effective dense retract domain representation of Cantor space.*

Definition 4.2. A *Cantor domain representation* of a space X is a domain representation $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \rho)$ where the domain is the Cantor domain.

Cantor domain representations exist for many spaces where, obviously, the cardinality of the space cannot exceed that of the continuum. For a space X let $\mathbf{CRep}(X)$ denote the set of Cantor domain representations of X. Clearly,

$$\mathbf{CRep}(X) \subseteq \mathbf{DRep}_\omega(X) \subseteq \mathbf{DRep}(X)\,.$$

Cantor domain representations are not retract representations unless the space is zero-dimensional, i.e., have a clopen base.

Proposition 4.3. *If $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \rho, \eta)$ is a retract representation of X, then X must be zero-dimensional.*

Proof. Cantor space is zero-dimensional and so has a clopen basis $\mathcal{B}$. By continuity of η, $\{\eta^{-1}[B] : B \in \mathcal{B}\}$ is a clopen basis for X. So X is zero-dimensional. □

4.2 TTE representations

In TTE, there are two kinds of *naming systems* for a space X. A *notation* is an onto partial map from the discrete space Σ^* to X, and a *representation* (in TTE terminology) is an onto partial map from Σ^ω to X. TTE notations can be modelled as domain representations using the flat domain $\Sigma^*_\perp$. TTE representations can be modelled as Cantor domain representations, where $\mathcal{C}^{\mathrm{R}}$ is a subset of Σ^ω. Formally, we have the following lemmas.

Lemma 4.4. *TTE notations are in one-to-one correspondence with domain representations where the domain is $\Sigma^*_\perp$ and the totality on the domain does not contain $\perp$.*

Proof. A TTE notation of X, i.e., a surjective partial map $\nu : \Sigma^* \rightharpoonup X$, gives rise to a domain representation $(\Sigma^*_\perp, \operatorname{dom}\nu, \nu)$ of X.

A domain representation $(\Sigma^*_\perp, {\Sigma^*_\perp}^{\mathrm{R}}, \rho)$ of X where the totality does not contain $\perp$ implies that ρ is a TTE notation since ${\Sigma^*_\perp}^{\mathrm{R}}$ is a subset of Σ^*. □

Lemma 4.5. *Any TTE representation gives rise to a Cantor domain representation.*

Proof. A TTE representation $\delta : \Sigma^\omega \rightharpoonup X$ of X gives rise to the domain representation $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \delta)$, where $\mathcal{C}^{\mathrm{R}} = \operatorname{dom}\delta$. □

In TTE, an arbitrary choice of $\mathcal{C}^{\mathrm{R}}$ is not considered, so the notion of Cantor domain representations is wider than the notion of TTE representations although not more expressive, as is shown below. However, some spaces are easier to give Cantor domain representations than TTE representations, e.g., the Cantor domain itself is trivially represented by itself with the identity, but requires some encoding to make into a TTE representation.

Theorem 4.6. *The class of spaces that have TTE representations coincides with the class of spaces that have countably based domain representations.*

Proof. Trivially, any space that has a TTE representation has a domain representation by Lemma 4.5.

For the other direction consider a countably based domain representation $(D, D^{\mathrm{R}}, \rho)$ of a space X. We will construct a Cantor domain representation of D, and in turn a Cantor domain representation of X. Let $\alpha : \omega \to D_c$ be a numbering of D_c, and let $\Sigma = \mathbf{2} = \{0, 1\}$. Define an encoding function $\iota : \mathbb{N} \to \mathbf{2}^*$ by

$$\iota(n) = 01^n0 .$$

Define also $\gamma : \mathcal{C} \to \mathcal{P}(\mathbb{N})$ by

$$\gamma(s) = \{n : \iota(n) \text{ is a substring of } s\} ,$$

that is, γ collects all natural numbers n encoded in the sequence s.

Define the representation function $\delta : \mathcal{C} \to D_c$ by

$$\delta(s) = \bigsqcup \gamma(p) ,$$

where p is the longest prefix of s such that $\gamma(p)$ is directed. Clearly, δ is a monotone map, so it has a unique continuous extension to $\mathcal{C}$. Let $\mathcal{C}^{\mathrm{R}}$ be the sequences $s \in \mathbf{2}^\omega$ such that $\gamma(s)$ is directed. Then $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \delta)$ is a Cantor domain representation of D.

By composition we have that $(\mathcal{C}, \delta^{-1}[D^{\mathrm{R}}], \rho\delta)$ is a Cantor domain representation of X. ☐

TTE also comes with a notion of reduction between naming systems, these are continuous partial functions. We will see that these reductions correspond to (total) continuous reductions in our sense.

Definition 4.7. Let $\delta : \Sigma^\omega \rightharpoonup X$ and $\epsilon : \Sigma^\omega \rightharpoonup X$ be TTE representations of X. A function $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$ *reduces* δ to ϵ if $\delta(x) = \epsilon f(x)$ for all $x \in \operatorname{dom} \delta$.

Lemma 4.8. *If $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$ reduces δ to ϵ then $f' : \mathcal{C} \to \mathcal{C}$ reduces $(\mathcal{C}, \operatorname{dom} \delta, \delta)$ to $(\mathcal{C}, \operatorname{dom} \epsilon, \epsilon)$, where f' is defined for compact c by*

$$f'(c) = \sqcap f[\uparrow d \cap \operatorname{dom} \delta] ,$$

where d is the maximal prefix of c such that $\uparrow d \cap \operatorname{dom} \delta \neq \emptyset$.

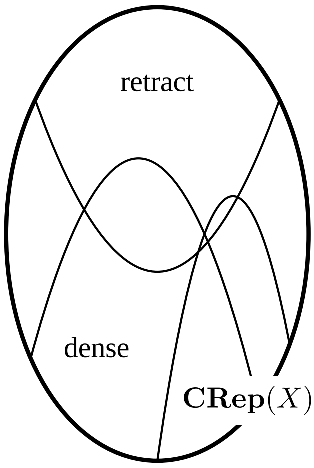
Proof. Note that the construction of f' is very similar to the second step of the construction used to lift functions to domain representations. Being restricted to the Cantor domain we can avoid the need to have $\operatorname{dom} \delta$ dense in $\mathcal{C}$ by carefully defining the function value for non-consistent approximations to be as small as possible while making f' monotone.

The monotone function f' has a unique continuous extension to all of $\mathcal{C}$. We leave to the reader the straightforward proof that $f'(x) = f(x)$ for all $x \in \operatorname{dom} \delta$. ☐

We leave to the reader to check that reductions between any naming systems (both notations and representations) corresponds to continuous (total) reductions in our sense.

The above implies that we can study TTE representations as the spectrum $\mathrm{Spec}(X, \mathbf{CRep}(X), \leq_c)$, which can be identified with a substructure of either

$\mathrm{Spec}(X, \mathbf{DRep}_\omega(X), \leq_{\mathrm{c}})$ or $\mathrm{Spec}(X, \mathbf{DRep}_\omega(X), \leq_{\mathrm{cp}})$. The spectrum using continuous reductions can roughly be drawn as follows.



4.3 Cantor–Weihrauch domain representations

We will now consider a standard construction of Cantor domain representations. For this section we fix the underlying alphabet Σ of Cantor space and the Cantor domain $\mathcal{C}$ to be $\mathbf{2} = \{0, 1\}$.

The following definition is due to Weihrauch except that we use a numbering of the subbase (from the natural numbers) rather than a notation (from the set of finite words).

Definition 4.9. An *effective topological space* is a triple $S = (X, \sigma, \alpha)$, where X is a non-empty T_0 space, σ is a countable subbase for X, and $\alpha : \omega \to \sigma$ is a semicomputable numbering of the topology.

Let $S = (X, \sigma, \alpha)$ be an effective topological space.

As in the proof of Theorem 4.6 define $\iota : \mathbb{N} \to \mathbf{2}^*$ and $\gamma : \mathcal{C} \to \mathcal{P}(\mathbb{N})$ by

$$\iota(n) = 01^n0$$

and

$$\gamma(s) = \{n : \iota(n) \text{ is a substring of } s\} .$$

Define a partial function $\delta_S : \mathcal{C} \to X$ by

$$\delta_S(s) = x, \text{ if } \gamma(s) = \{n : x \in \alpha(n)\} .$$

The function δ_S above is well-defined since the topology was assumed to be T_0. The objects for which δ_S is defined are sequences encoding all atomic properties of some point via the numbering α of the subbase.

The following notion corresponds to the notion of *standard representation* in [34].

Definition 4.10. A *Cantor–Weihrauch domain representation (CW domain representation)* is the Cantor domain representation $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \delta_S)$, where $\mathcal{C}^{\mathrm{R}} = \operatorname{dom} \delta_S$.

CW domain representations are not dense if the space X contains more than one point.

Proposition 4.11. *A CW domain representation is effective.*

Proof. The Cantor domain is an effective domain. □

4.4 Continuous Reductions

We look here at where CW domain representations belong in the spectrum $\operatorname{Spec}(X, \mathbf{DRep}(X), \leq_{\mathrm{c}})$ of domain representations under continuous reductions. In fact, CW domain representations are below a naturally constructed dense retract domain representation. In general retract representation does not reduce to CW domain representations by Proposition 4.3.

Let $S = (X, \sigma, \alpha)$ be an effective topological space, and let

$$P = \{X\} \cup \left\{ \bigcap A : A \in \mathcal{P}_{\mathrm{f}}(\sigma),\ \bigcap A \neq \emptyset \right\} .$$

Thus, P is a base for the topology τ on X. Ordered by reverse inclusion P is a neighbourhood system, and by Theorem 5.4 of [6] the ideal completion over this neighbourhood system is a dense retract representation $(D, \eta[X], \eta^{-1})$ of X.

Proposition 4.12. *The representation* $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \delta_S)$ *is reducible to* $(D, \eta[X], \eta^{-1})$.

Proof. Let $c \in \mathcal{C}_{\mathrm{c}}$, that is, c is a finite sequence over $\mathbf{2}$. Define $\phi : \mathcal{C}_{\mathrm{c}} \to D$ by

$$\phi(c) = \bigcap \{\alpha(n) : \iota(n) \text{ is a substring of } p\} ,$$

where p is the longest prefix of c for which the intersection is non-empty. The monotone function ϕ can uniquely be extended to a continuous function $\phi : \mathcal{C} \to D$.

If $s \in \mathcal{C}^{\mathrm{R}}$ and $\delta_S(s) = x$, then $\phi(s)$ is the supremum of basic open sets all of which contain x, so $\phi(s) \sqsubseteq I_x = \{A \in P : x \in A\}$. Any basic open set U containing x is the intersection of a finite set of subbasic open sets and any such finite set of subbasic open sets are encoded into s so $U \sqsubseteq \phi(s)$. Showing that $\phi(s) = I_x$. Thus, $\phi[\mathcal{C}^{\mathrm{R}}] \subseteq \eta[X]$ and $\eta^{-1}\phi = \delta_S$. □

5 Admissible representations

5.1 Notions of Admissibility

We will here consider some of the notions of admissibility considered for (TTE and domain) representations. We will often restrict our attention to domain representations that are limited in size in the sense that the set of compact elements have bounded cardinality.

The classical notion of admissibility in TTE is the following notion due to Weihrauch [34].

Definition 5.1. Let X be a second countable T_0 space. A domain representation $D \in \mathbf{DRep}_\omega(X)$ is *W-admissible* if D is continuously equivalent to a CW domain representation of X.

The original definition of Weihrauch is formulated in the less general setting of representations in $\mathbf{CRep}(X)$, but immediately generalises to the superclass $\mathbf{DRep}_\omega(X)$.

The notion of W-admissibility only applies to spaces that have CW domain representations and these are the second countable T_0 spaces. Schröder [23] considers an extended notion of admissibility that includes some spaces that are not second countable.

Definition 5.2. A Cantor domain representation $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \rho)$ of X is *S-admissible* if any continuous partial function $\epsilon : \Sigma^\omega \rightharpoonup X$ factors through D, i.e., if there exists $\phi : \Sigma^\omega \rightharpoonup \Sigma^\omega$ such that $\epsilon = \rho\phi$ on $\operatorname{dom} \epsilon$.

Note, that ϵ in the definition is not assumed to be onto X.

Hamrin [18] generalised the above notion of admissibility to domain representations.

Definition 5.3. (i) A domain representation D is *κ-admissible* if for each κ-based domain E with dense totality E^{R} and for each continuous function $\epsilon : E^{\mathrm{R}} \to X$ there exists a continuous function $\bar{\epsilon} : E \to D$ such that $\epsilon(x) = \rho\bar{\epsilon}(x)$ for all $x \in E^{\mathrm{R}}$.

(ii) A domain representation is *H-admissible* if it is κ-admissible for all κ.

As for S-admissibility, note that the function ϵ in the definition above is not assumed to be a representation function, in particular, it need not be onto.

5.2 Admissibility as Universality

We will see here that the notions of admissibility due to Schröder and Hamrin are all notions of universality in the appropriate spectrum. Recall also that universality can be seen as being a terminal object in the appropriate category as observed in Section 3.1.

W-admissibility is not intrinsically a universality notion as it is defined differently, but W-admissible representations are universal in $\mathrm{Spec}(X, \mathbf{CRep}(X), \leq_{\mathrm{c}})$ as they are continuously equivalent to universal CW domain representations.

The other notions of admissibility are nearly formulated as universality conditions already, but there is a small difference in that universality requires just representations to factor through the universal representation, but admissible representations require that all continuous maps to the space should factor through the representation (these continuous maps could be viewed as *partial representations* of the space). Thus, we have something to prove for each of the admissibility notions considered. The essential step in these proofs is a disjoint sum construction.

Proposition 5.4. *A domain representation* $(D, D^{\mathrm{R}}, \rho)$ *in* $\mathbf{DRep}_\kappa^{\mathrm{D}}(X)$ *is* κ*-admissible if, and only if, it is universal in* $\mathbf{DRep}_\kappa^{\mathrm{D}}(X)$.

Proof. ($\Rightarrow$): All domain representations in $\mathbf{DRep}_\kappa^{\mathrm{D}}$ reduces to a κ-admissible representation D by definition of κ-admissibility, so D is universal.

($\Leftarrow$): Let E a κ-based domain with a dense totality E^{R}. We need to show that any continuous $\epsilon : E^{\mathrm{R}} \to X$ factors through D.

Construct $F = E \oplus D$, the disjoint union of E and D. Clearly, F is κ-based. The totality $F^{\mathrm{R}} = E^{\mathrm{R}} \cup D^{\mathrm{R}}$ is dense. Define $\rho_F : F^{\mathrm{R}} \to X$ by

$$\rho_F(f) = \begin{cases} \epsilon(f), & \text{if } f \in E^{\mathrm{R}}; \\ \rho_D(f), & \text{otherwise}\,. \end{cases}$$

Continuity of ρ_F follows from the continuity of ϵ and ρ_D on the disjoint spaces E^{R} and D^{R}. Thus, $(F, F^{\mathrm{R}}, \rho_F)$ is a dense κ-based domain representation of X. Since D is universal there exists a continuous $\phi : F \to D$ representing the identity on X.

Let $\iota : E \to F$ be the continuous embedding of E into F. Let $\bar{\epsilon} = \phi\iota : E \to D$. By construction we have $\rho_D\bar{\epsilon} = \rho_D\phi\iota = \epsilon$ showing that ϵ factors through D. ☐

Proposition 5.5. *A domain representation* $(D, D^{\mathrm{R}}, \rho)$ *in* $\mathbf{DRep}^{\mathrm{D}}(X)$ *is H-admissible if, and only if, it is universal in* $\mathbf{DRep}^{\mathrm{D}}(X)$.

Proof. Idem as above without cardinality considerations. ☐

Proposition 5.6. *A Cantor domain representation* $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \rho)$ *is S-admissible if, and only if, it is universal in* $\mathbf{CRep}(X)$.

Proof. Assume without loss of generality that the underlying alphabet of Cantor spaces contains the symbols 0 and 1. The disjoint sum of two Cantor domain representations $(\mathcal{C}, \mathcal{C}_i^{\mathrm{R}}, \rho_i)$, $i = 0, 1$ is the Cantor domain representation $(\mathcal{C}, \mathcal{C}^{\mathrm{R}}, \rho)$ where

$$\mathcal{C}^{\mathrm{R}} = \{0u : u \in \mathcal{C}_0^{\mathrm{R}}\} \cup \{1u : u \in \mathcal{C}_1^{\mathrm{R}}\}\,,$$

and

$$\rho(u) = \begin{cases} \rho_0(v), & \text{if } u = 0v; \\ \rho_1(v), & \text{if } u = 1v\,. \end{cases}$$

The rest of the proof is identical to the proofs above. ☐

It is known that W-admissibility implies S-admissibility within $\mathbf{CRep}$, and trivially we have that H-admissibility implies κ-admissibility for any κ.

By characterisation results of Schröder [23, Theorem 13] and Hamrin [18, Theorem 6.8] we have that spaces have S-admissible representations if, and only if, they have ω-admissible representations. We would like to relate S-admissibility and ω-admissibility directly using reductions, which would be easy if we had a natural class of domain representations that contains both $\mathbf{CRep}$ and $\mathbf{DRep}_\omega^{\mathrm{D}}$. The obvious choice would be $\mathbf{DRep}_\omega$, but representations universal in $\mathbf{DRep}_\omega^{\mathrm{D}}(X)$ need not be universal in the superclass $\mathbf{DRep}_\omega(X)$.

6 Representations of the reals

Here we will look at three different representations of real numbers. The first is the customary interval domain, the second is a substructure of the interval domain that allow for more efficient computations, and the third corresponds to binary expansion of the reals.

Let $\mathcal{R}$ be the ideal completion of all closed rational intervals together with the real line ordered by reverse inclusion. The representing elements $\mathcal{R}^{\mathrm{R}}$ of this domain are all ideals that have singleton intersections; a representing ideal is mapped by $\rho_{\mathcal{R}}$ to the single element of its intersection. Define $\eta_{\mathcal{R}}$ by

$$\eta_{\mathcal{R}}(x) = \{[a,b] : a < x < b,\ a,b \in \mathbb{Q}\} .$$

Lemma 6.1. *$(\mathcal{R}, \mathcal{R}^{\mathrm{R}}, \rho_{\mathcal{R}}, \eta_{\mathcal{R}})$ is an admissible representation of the reals.*

Proof. A standard proof shows that the representation is a dense retract domain representation. □

In [8] centred dyadic approximations are considered for efficient implementations of exact real arithmetic. These form an interesting substructure of the interval domain.

Definition 6.2. A *centred dyadic interval* is represented by a triple (m, e, s) of the form

$$a = (m \pm e)2^{-s} ,$$

where the *mantissa* m and the *exponent* s are integers, and the *error term* e is a natural number. A real x is *approximated* by a if

$$|x - m2^{-s}| \leq e2^{-s} ,$$

or equivalently,

$$x \in [(m-e)2^{-s}, (m+e)2^{-s}] .$$

Fix $j > 0$. A *centred dyadic j-approximation* is a centred dyadic interval where the error term is strictly bounded by 2^j.

We will assume that j is fixed throughout and we will simply write *centred dyadic approximation*.

Let $\mathcal{R}_{\mathrm{cda}}$ be the ideal completion of all centred dyadic approximations together with the real line ordered by reverse inclusion. Representing elements $\mathcal{R}_{\mathrm{cda}}^{\mathrm{R}}$ are again ideals with singleton intersection and the representing function $\rho_{\mathcal{R}_{\mathrm{cda}}}$ is defined as before.

It is shown in [8, Lemma 3.7] that $\mathcal{R}_{\mathrm{cda}}$ is not a domain, but that it is a bifinite domain (or SFP-domain). Nevertheless we will show that with respect to reducibility $\mathcal{R}_{\mathrm{cda}}$ is equivalent to the interval domain. The following lemma shows that even though finite suprema does not exist in general in $\mathcal{R}_{\mathrm{cda}}$ there is a sufficiently rich substructure of $\mathcal{R}_{\mathrm{cda}}$ where finite suprema exist.

Lemma 6.3. *Within the substructure of all centred dyadic 1-approximations finite suprema exist.*

Proof. It is sufficient to show that the supremum of $a = (m \pm 1)2^{-s}$ and $b = (n \pm 1)2^{-t}$ is a centred dyadic 1-approximation. Assume without loss of generality that $t \geq s$.

If the distance between the centre points of a and b, that is $m2^{-s}$ and $n2^{-t}$, is less than the radius of a, that is 2^{-s}, then the centre of b must be at least 2^{-t} away from boundary of a, meaning that b is contained in a, so b is the supremum.

The remaining case is that the centre of b is on the boundary of a. Assume that the centre of b is the upper end-point of a, i.e., $n2^{-t} = (m+1)2^{-s}$. The supremum of a and b is $((2n-1) \pm 1)2^{-t-1}$. □

Theorem 6.4. *The representations $\mathcal{R}$ and $\mathcal{R}_{\mathrm{cda}}$ are equivalent.*

Proof. The inclusion map from $\mathcal{R}_{\mathrm{cda}}$ to $\mathcal{R}$ represents the identity on the real line so $\mathcal{R}_{\mathrm{cda}}$ reduces to $\mathcal{R}$.

For the other direction define ϕ on compacts by

$$\phi([a,b]) = \bigsqcup_1 \{(m \pm 1)2^{-s} : [a,b] \subseteq (m \pm 1)2^{-s},\ 2^{-s} \leq |b-a|\},$$

if $a \neq b$ and by

$$\phi([a,a]) = \bigsqcup \{(m \pm 1)2^{-s} : a \text{ is approximated by } (m \pm 1)2^{-s}\},$$

otherwise. The $\bigsqcup_1$ in the former equation gives finite suprema in the substructure of centred dyadic 1-approximations. Note that the supremum is taken over a finite set because there can only be finitely many 1-approximations containing the interval when the radii of the 1-approximations are bounded. The second supremum is taken over a directed set.

Extend ϕ to a continuous function. Then ϕ represents the identity on the real line. Thus, $\mathcal{R}$ reduces to $\mathcal{R}_{\mathrm{cda}}$. □

Restricting the interval domain $\mathcal{R}$ to the unit interval gives an admissible representation of the unit interval which we denote by $\mathcal{R}_{[0,1]}$.

We will construct a representation corresponding to binary expansion. For simplicity we restrict ourselves to the unit interval. Let the compact elements be finite binary expansions, and let $D_{[0,1]}$ be the ideal completion over the prefix ordering of finite binary expansions. A maximal element of the domain corresponds to an infinite binary expansion. Let $\rho_D : D_{[0,1]} \to \mathbb{R}$ be the mapping of an infinite binary expansion into the corresponding real number. Note that any interior dyadic point will have two representations in the domain, for example, $\frac{1}{2}$ is represented by both $.0111\ldots$ and $.1000\ldots$. The domain $D_{[0,1]}$ is a dense representation of the unit interval.

Construct a monotone map ϕ from compact elements of $D_{[0,1]}$ (that is, finite binary expansions) to $\mathcal{R}_{[0,1]}$ by mapping $.b_1 \ldots b_n$ to the interval

$$\left[\frac{a}{2^n}, \frac{a+1}{2^n}\right],$$

where a is the integer number $b_1 \ldots b_n$. Extend ϕ to a continuous function $\phi : D_{[0,1]} \to \mathcal{R}_{[0,1]}$. The function ϕ induces the identity on the unit interval so $D_{[0,1]}$ reduces to $\mathcal{R}_{[0,1]}$.

Lemma 6.5. *The representation $\mathcal{R}_{[0,1]}$ does not reduce to $D_{[0,1]}$.*

Proof. Assume that $\mathcal{R}_{[0,1]}$ reduces to $D_{[0,1]}$. Then by Lemma 3.11 $D_{[0,1]}$ would be a retract representation. Consider where the embedding would send $\frac{1}{2}$. If the embedding of $\frac{1}{2}$ is to $.0111\ldots$ then the preimage of the basic open set $\uparrow .0$ is the non-open interval $[0, \frac{1}{2}]$. If $\frac{1}{2}$ instead is embedded into $.1000\ldots$ then the pre-image of $\uparrow .1$ is the non-open interval $[\frac{1}{2}, 1]$. This contradicts the existence of a continuous embedding, and hence, that $D_{[0,1]}$ is a retract representation. Thus, $\mathcal{R}_{[0,1]}$ does not reduce to $D_{[0,1]}$. □

Indeed, $D_{[0,1]}$ is not an admissible domain representation nor is its corresponding TTE representation S-admissible.

We also know that representing the reals by binary expansions is not an appropriate choice when considering computability of operations on the domain. It is well-known that neither addition nor multiplication is computable on binary expansions of real numbers.

Example 6.6. Consider computing addition on infinite binary expansions. The sum of $\frac{1}{3} = 0.010101\ldots$ and $\frac{1}{6} = 0.0010101\ldots$ is $\frac{1}{2}$. However, for whatever finite amount of the inputs that is inspected even the first bit after the binary point of the output is not determined. Thus, we cannot effectively compute addition.

Theorem 6.7. *The cardinality of $\mathrm{Spec}(\mathbb{R}, \mathbf{DRep}(\mathbb{R}), \leq_c)$ is infinite.*

Proof. Let $b \geq 2$ be a base. Let D_b be the domain representation of the reals using base b expansions of the reals. Let p be a prime such that p does not divide b. Then the fraction $\frac{1}{p}$ has an infinite expansion in base b. There is no continuous reduction of D_b to D_p since any finite prefix of the expansion of $\frac{1}{p}$ in D_b is not enough to determine if it should be sent to $.0\ldots$ or $.1\ldots$ in D_p. The result follows as there are infinitely many prime numbers. □

As the proof shows, there are infinitely many non-equivalent countably based domain representations of the reals.

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