Fixed-Point-Free is NP-complete

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The problem FPF

- Ω: a finite set $\{1, 2, \cdots, n\}$
- G: a permutation group on Ω, i.e, a subgroup of $sym(\Omega)$ (or S_n). Assume G is given via a set of generators.
- the natural action $\Omega \times G \to \Omega$.
- an element $g \in G$ is called fixed point free (derangement) if $\mathrm{fix}_\Omega(g) = \{\alpha \in \Omega | \alpha g = \alpha\} = \emptyset$.
- all such elements form a subset of G, denoted by: $\mathrm{FPF}(G) = \{g \in G | \mathrm{fix}_\Omega(g) = \emptyset\}$.
- Question: Given any G acting on the set Ω, is $\mathrm{FPF}(G) = \emptyset$?

Example

$\Omega = \{1, 2, \cdots, 6\}$

- $\mathrm{FPF}(G) \neq \emptyset$:
  - Trivial Case: Transitive action (orbit number is 1) via orbit-counting lemma.
  - Nontrivial Case:
    $G = \{id, (1\ 2\ 3), (4\ 5\ 6), (1\ 2\ 3)(4\ 5\ 6)\}$.
- $\mathrm{FPF}(G) = \emptyset$:
  - Trivial case: some orbit is of size 1;
  - Nontrivial case:
    $G = \{id, (1\ 2)(3\ 4), (1\ 2)(5\ 6), (3\ 4)(5\ 6)\}$.

An NP-complete problem: NAESAT

- **NP**-complete problem: 3-**SAT**
  - a finite set of boolean variables U;
  - a collection of clause $C = \{c_1, \cdots, c_m\}$, each has the form $c_k = a_{k,1} \vee a_{k,2} \vee a_{k,3}$ where $a_{k,i} = u_i$ or $\bar{u}_i$.
  - Question: Is there a satisfying truth assignment for all clauses.
  - Example: $U = \{u_1, u_2, u_3\}$; $c_1 = u_1 \vee u_2 \vee u_3$, $c_2 = \bar{u}_1 \vee u_2 \vee u_3$.
- **NAESAT**: In no clauses are all three literals equal in truth value.
- **NAESAT** is **NP**-complete.

Reduction 1: Gadgets

- The Variable Gadget: the cycle $t_i = (2i-1, 2i)$ for each variable u_i.
- The Clause Gadget: Assume $c_k = a_{k,1} \vee a_{k,2} \vee a_{k,3}$.
  - $d_{k,1} = (d+1, d+2)(d+3, d+4)$ to the element $a_{k,1}$
  - $d_{k,2} = (d+1, d+3)(d+2, d+4)$ to the element $a_{k,2}$
  - $d_{k,3} = (d+1, d+4)(d+2, d+3)$ to the element $a_{k,3}$

  where $d = 2n + 4(k-1)$.

Reduction 2: Construction

- $G =< g_1, g_1', \cdots, g_n, g_n' >$ Where the cycles for each generator is given as follows:
  - t_i to g_i, g_i'.
  - $d_{k,j}$ to g_i if $a_{k,j} = u_i$
  - $d_{k,j}$ to g_i' if $a_{k,j} = \bar{u}_i$
- For an instance of **NAESAT** with n variables and m clauses, we have $|\Omega| = 2n + 4m$, and G having $2n$ generators. This means the reduction is polynomial.
- $\mathrm{FPF}(G) \neq \emptyset$ iff the corresponding **NAESAT** is satisfiable.

Example

- **NAESAT**: $U = \{u_1, u_2, u_3\}$; $c_1 = u_1 \vee u_2 \vee u_3$, $c_2 = \bar{u}_1 \vee u_2 \vee u_3$.
- **FPF**: $G =< g_1, g_1', g_2, g_2', g_3, g_3' >$ where
  - $g_1 = (1\ 2)(7\ 8)(9\ 10)$
  - $g_1' = (1\ 2)(11\ 12)(13\ 14)$
  - $g_2 = (3\ 4)(7\ 9)(8\ 10)(11\ 13)(12\ 14)$
  - $g_2' = (3\ 4)$
  - $g_3 = (5\ 6)(7\ 10)(8\ 9)(11\ 14)(12\ 13)$
  - $g_3' = (5\ 6)$
-
  - satisfying assignments: {(T,T,F),(T,F,T),(F,T,F),(F,F,T)}
  - $\mathrm{FPF}(G) = \{g_1 g_2 g_3', g_1 g_2' g_3, g_1' g_2 g_3', g_1' g_2' g_3\}$
  - The reduction is 1-1.

Metrics on Permutations

- Def:Hamming Distance $H(\pi, \sigma) = \#\{i : \pi(i) \neq \sigma(i)\}$.
- Def: Hamming Weight $W_H(\pi) = H(\pi, Id)$.
- Fact:$\mathrm{FPF}(G) = \{\pi : |W_H(\pi) = n\}$.
- Many other distances: Cayley, Ulam, Footrule, Kendall's Tau, Spearman's rank correlation......
- Given a distance on permutation group, the corresponding Max(Min) Weight problem is to decide the Max.(Min.) weight.

Conclusions

- $\mathbf{FPF}$ is NP-complete. (independently by C.Buchheim & M.Jünger)
- $\#\mathbf{FPF}$ is #P-complete
- Max(Min)-Hamming weight problem is NP-hard. (also B & J)
- The corresponding Max(Min)-weight problem is:
  - NP-hard: Cayley,Ulam, Footrule, Kendall's Tau, Spearman's rank correlation.......
  - P: $l_\infty(\pi, \sigma) = \max_{1 \leq i \leq n} |\pi(i) - \sigma(i)|$......
- To decide the nontrivial Max(Min) Cycles is NP-hard.
- To decide the permutation character is NP-hard.

Open problems

- Characterize respectively the distance that lies in NP-hard and P.
- The complexity of counting $|\mathrm{FPF}(G)|$ when G is transitive, note now $\mathrm{FPF}(G) \neq \emptyset$ by orbit counting lemma.
- Approximately count $|\mathrm{FPF}(G)|$.
- Compute the irreducible character.