Hard-To-Solve Bimatrix Games

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Nash equilibria of bimatrix games

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} \qquad x^TB = \begin{bmatrix} 8/3 & 8/3 \end{bmatrix}$$

only **pure best responses** can have probability > 0

Best response polytope Q for player 2

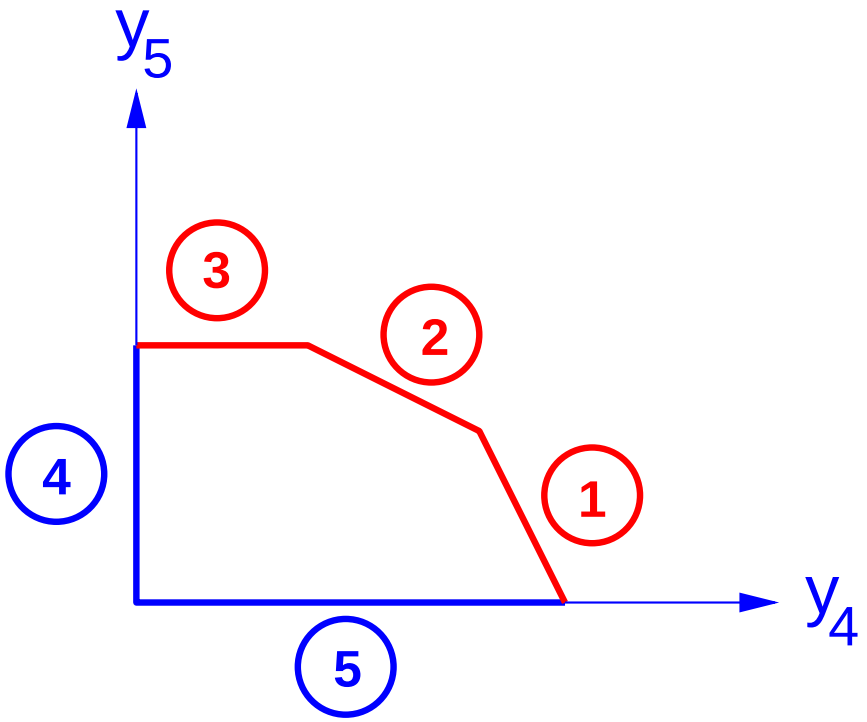
	y_4	y_5
①	3	3
②	2	5
③	0	6

$= A$

$$Q = \{ y \mid Ay \leq 1,\ y \geq 0 \}$$

$$Q = \{ (y_4, y_5) \mid$$

$$\begin{aligned}
① &: \quad 3y_4 + 3y_5 \leq 1 \\
② &: \quad 2y_4 + 5y_5 \leq 1 \\
③ &: \quad\quad 6y_5 \leq 1 \\
④ &: \quad y_4 \geq 0 \\
⑤ &: \quad\quad y_5 \geq 0 \ \}
\end{aligned}$$



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\\
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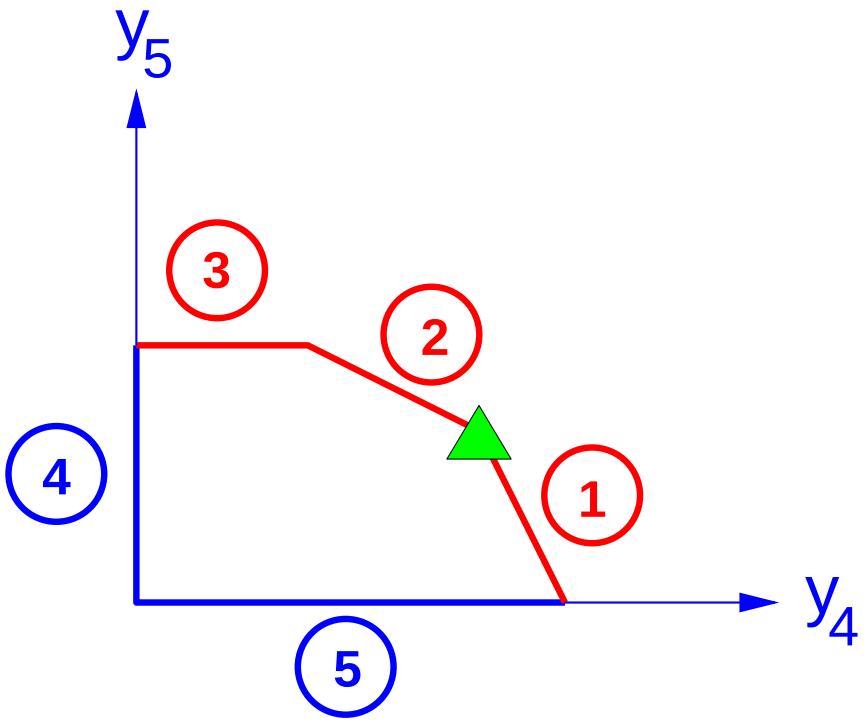
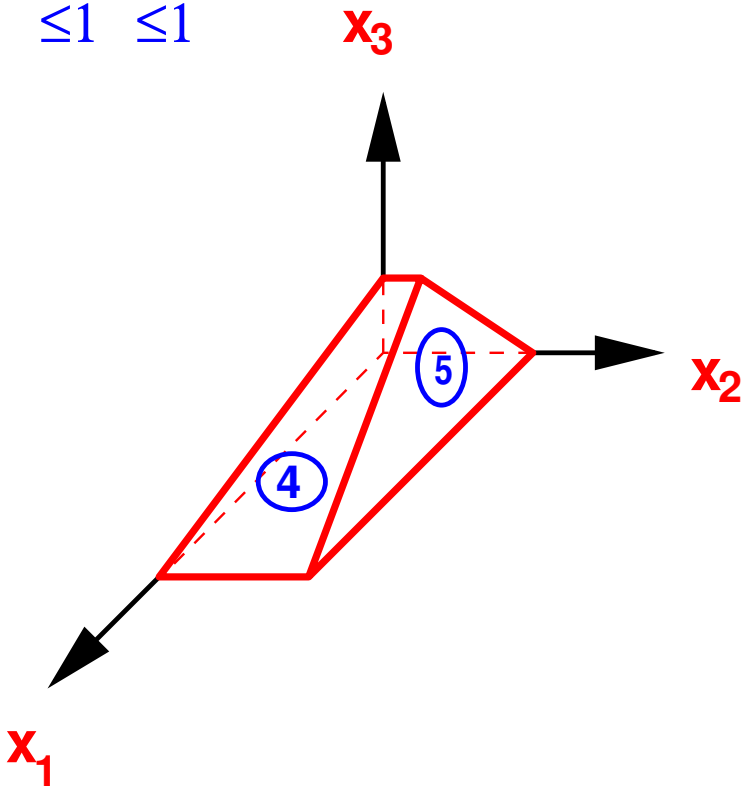


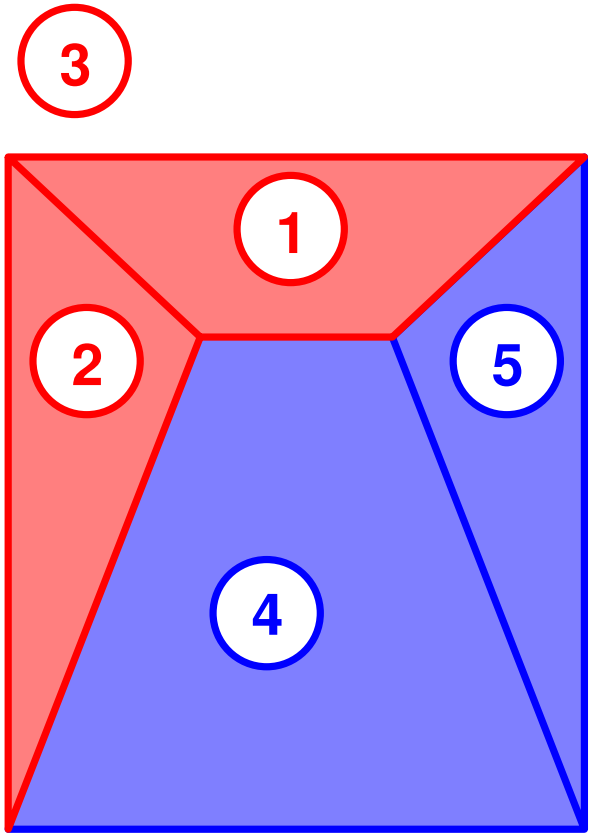
Best response polytope P for player 1

x_1	1	0
x_2	0	2
x_3	4	3
	≤1	≤1

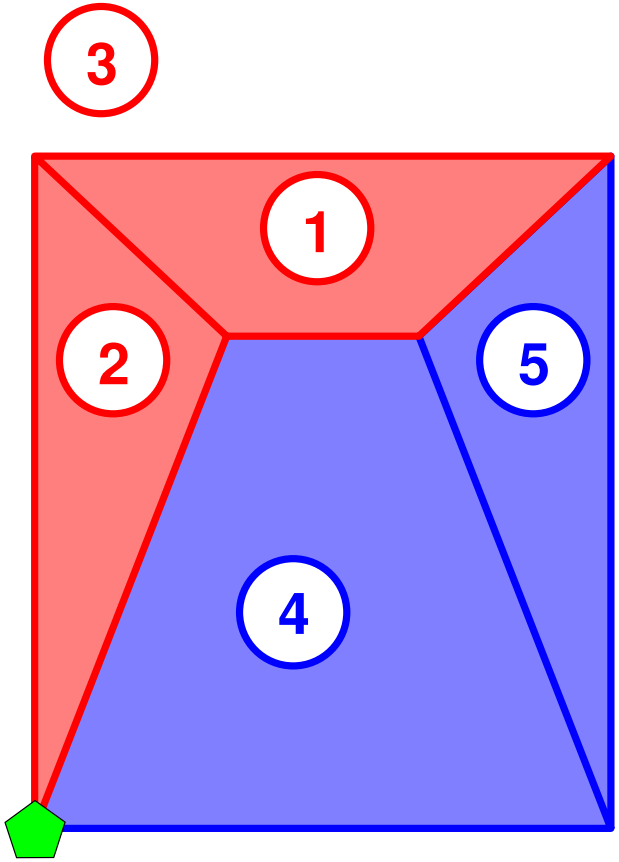
= B

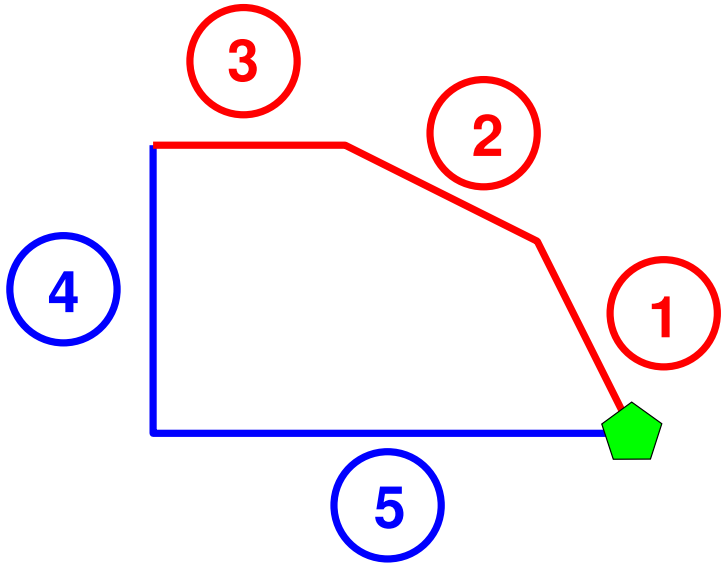
$$P = \{ x \mid x \geq 0,\ x^{\top} B \leq 1 \}$$





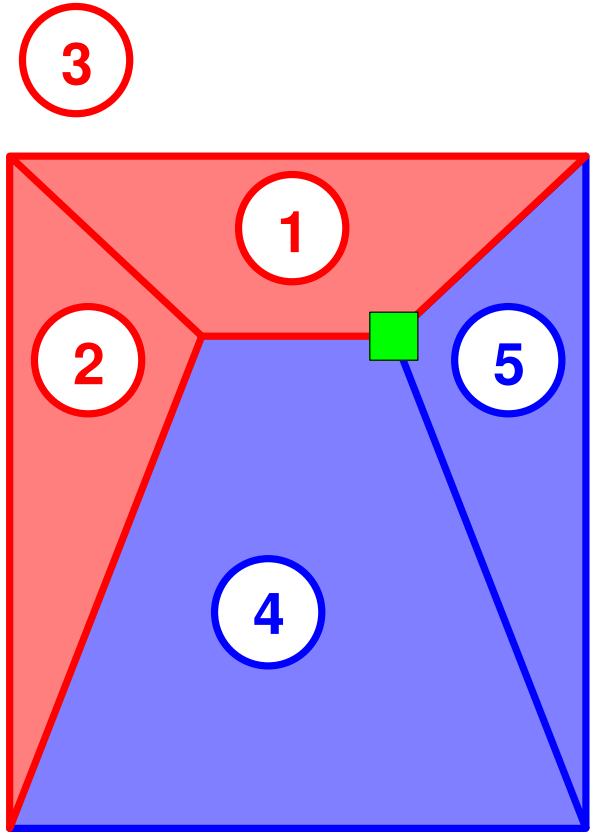
Equilibrium = completely labeled pair

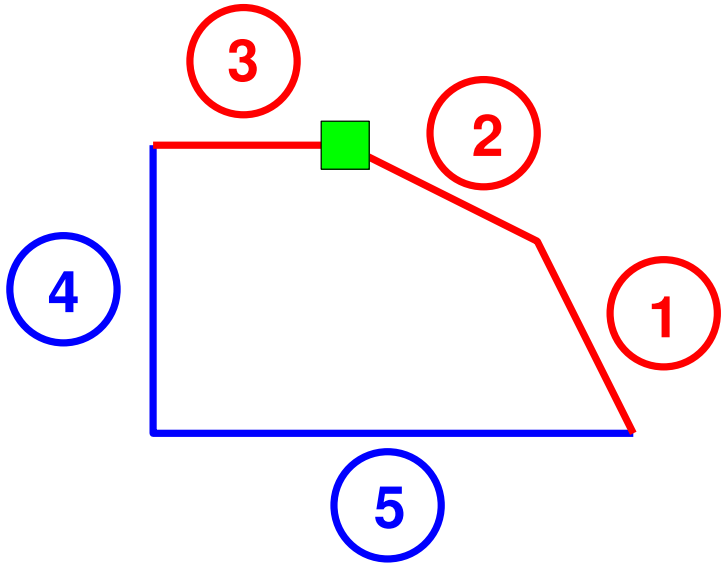




pure equilibrium

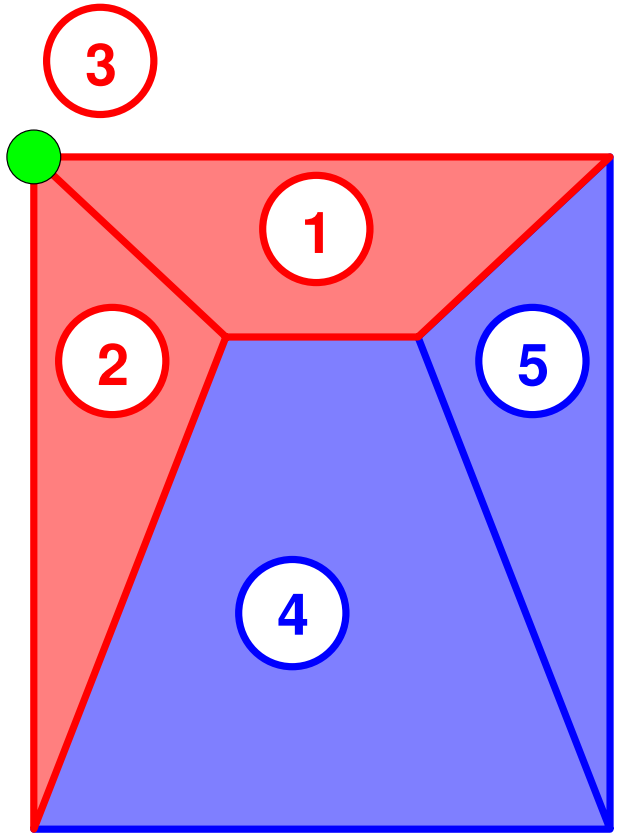
Equilibrium = completely labeled pair

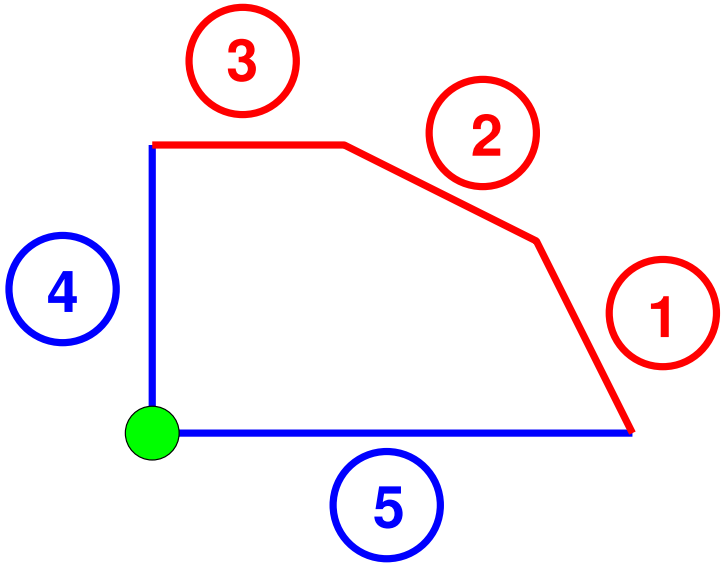




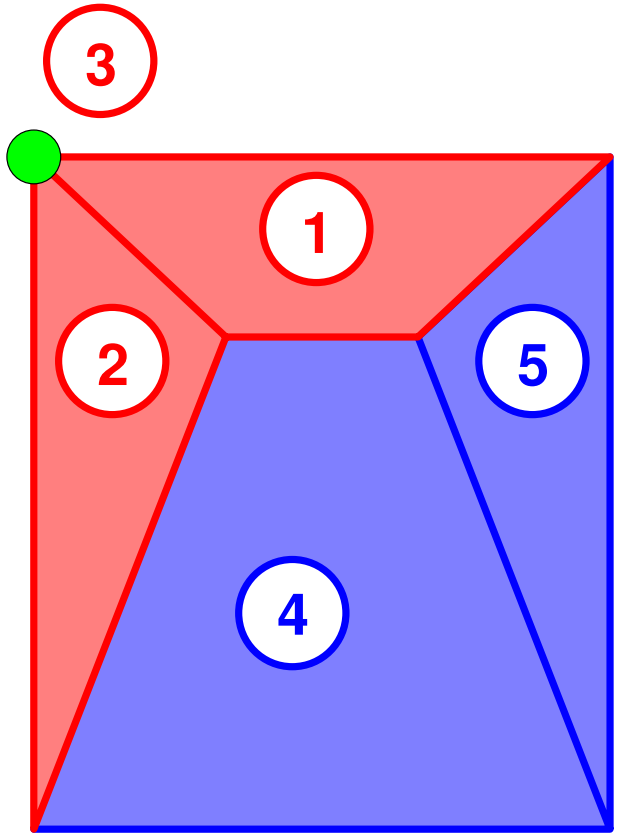
mixed equilibrium

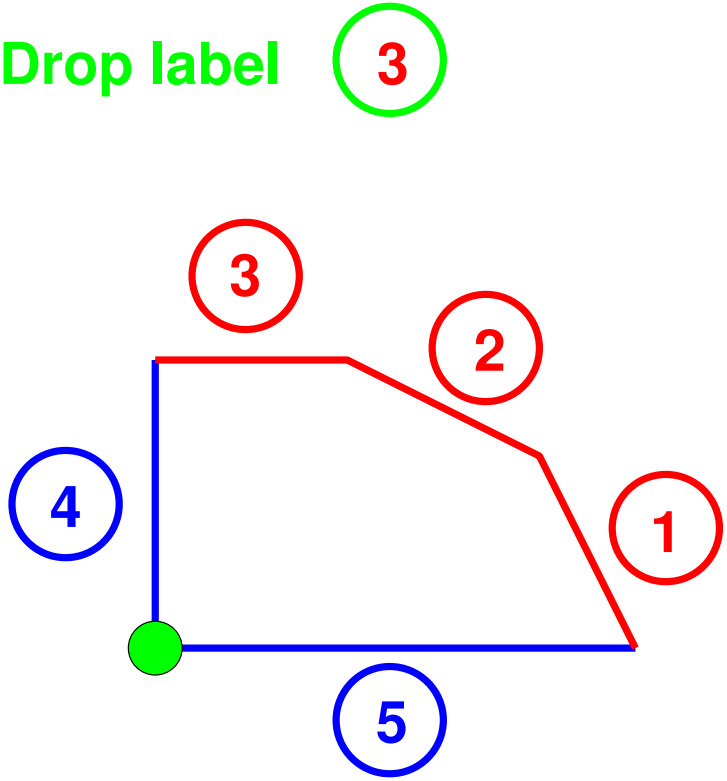
The Lemke–Howson algorithm





The Lemke–Howson algorithm





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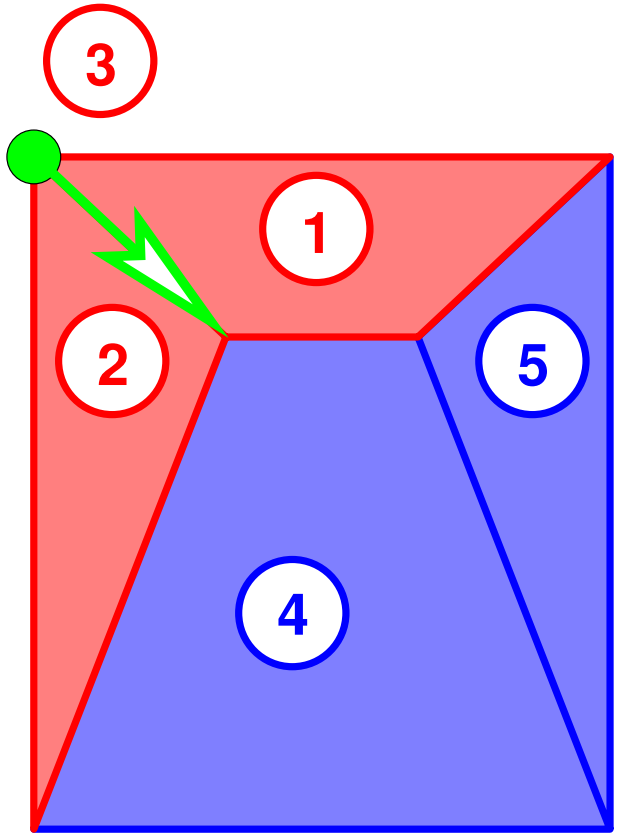

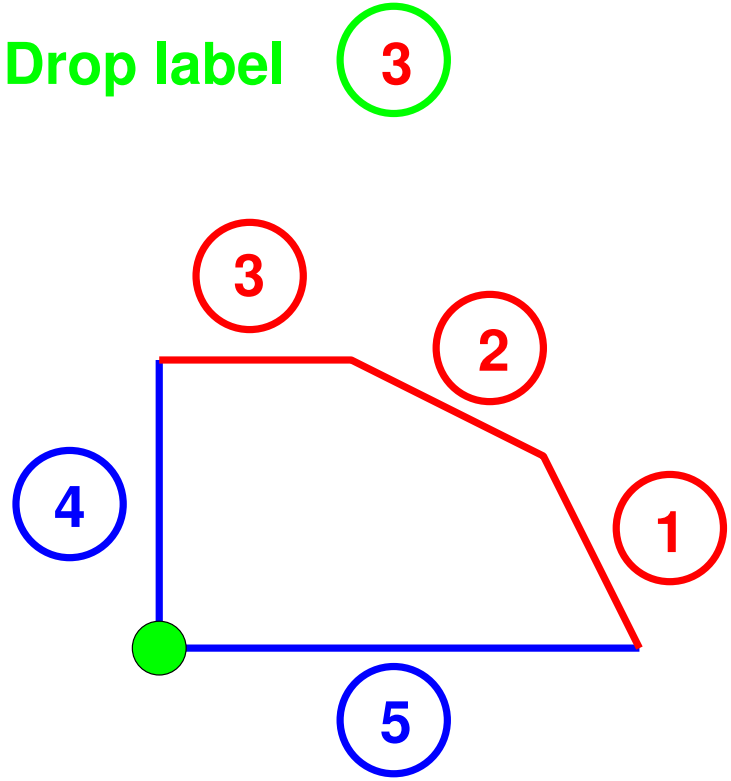

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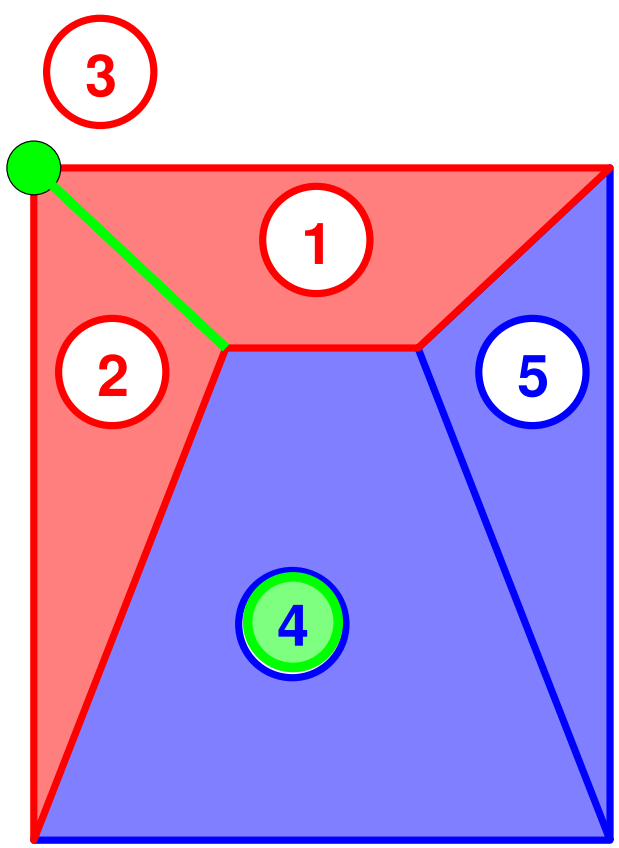

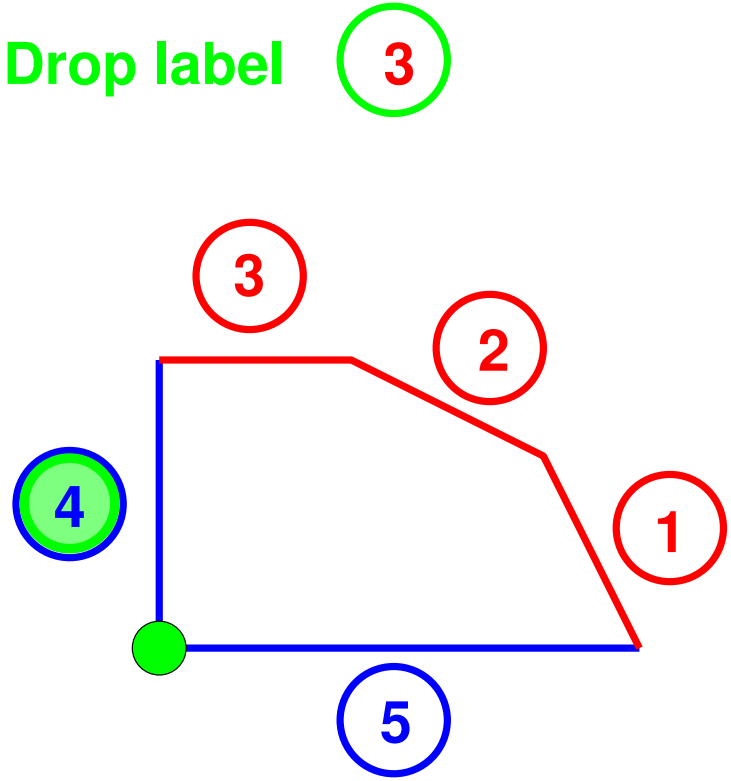

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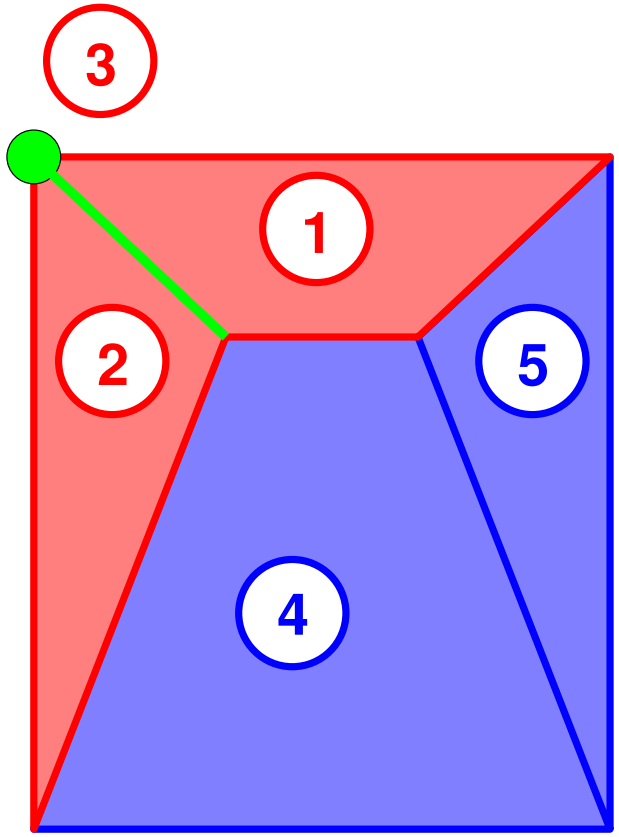

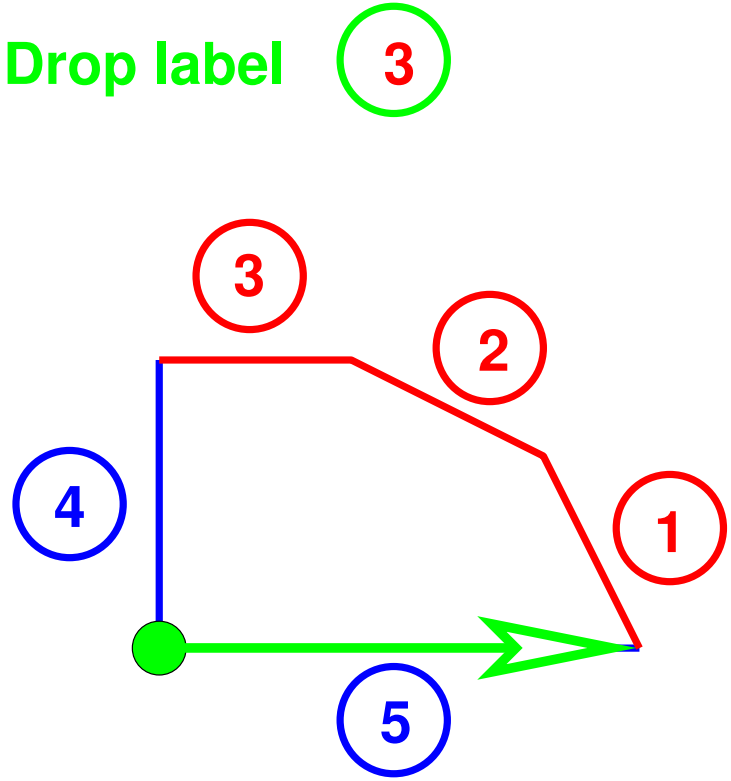

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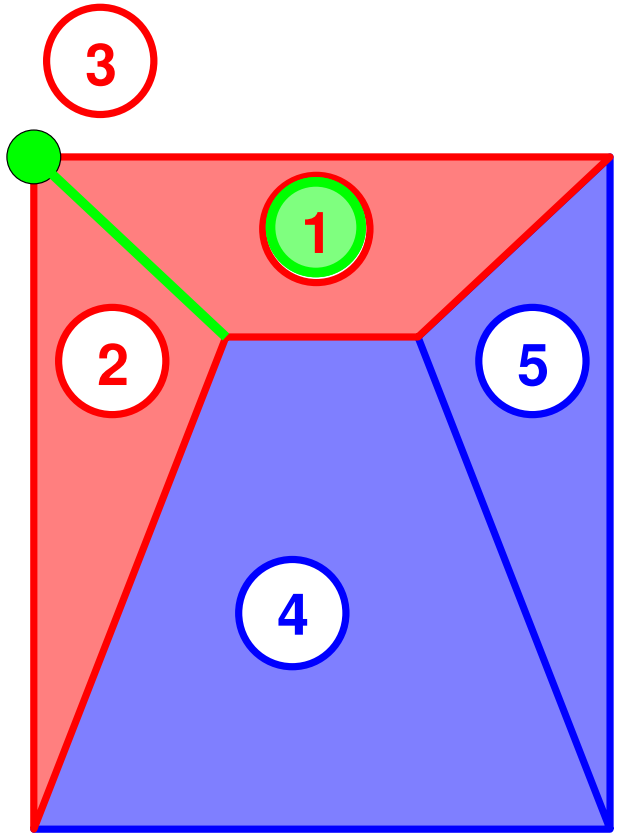

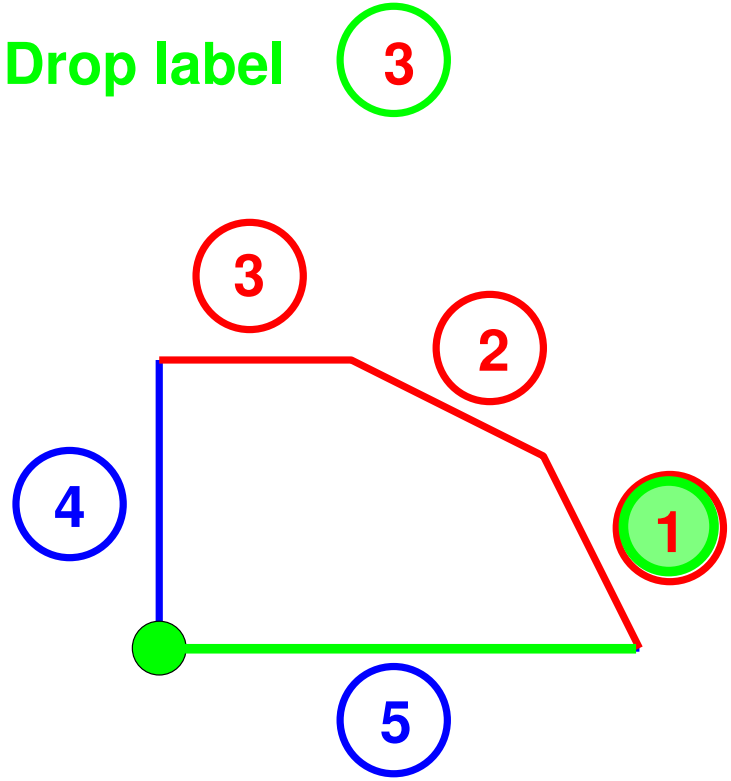

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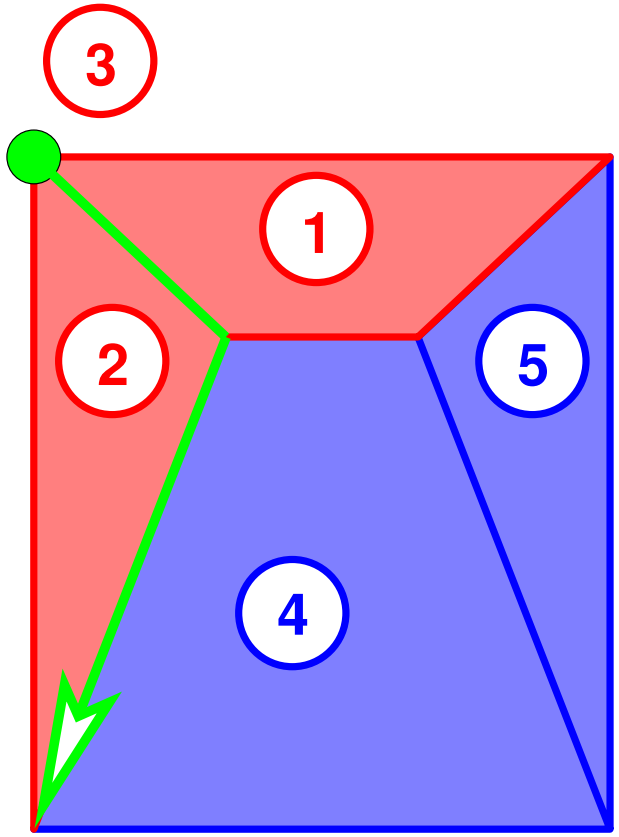

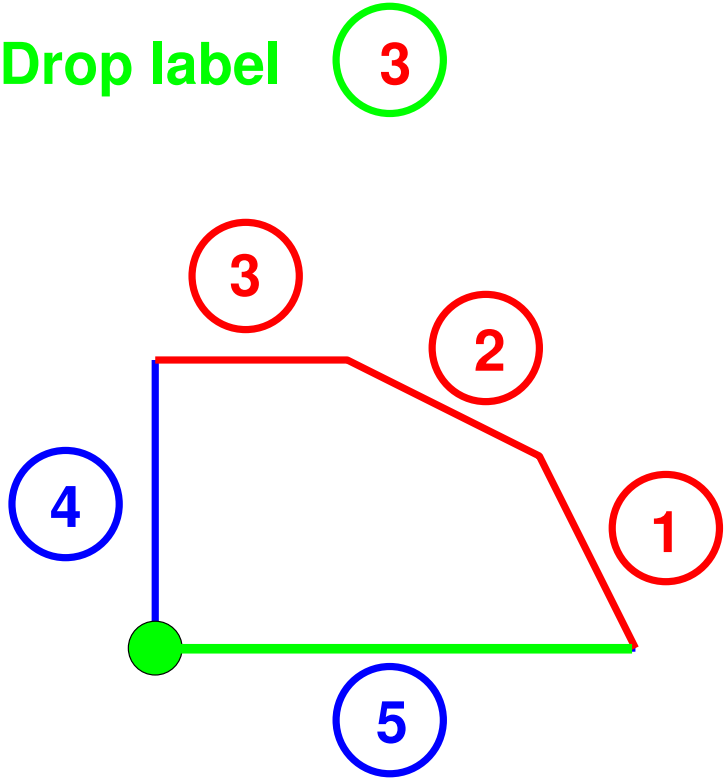

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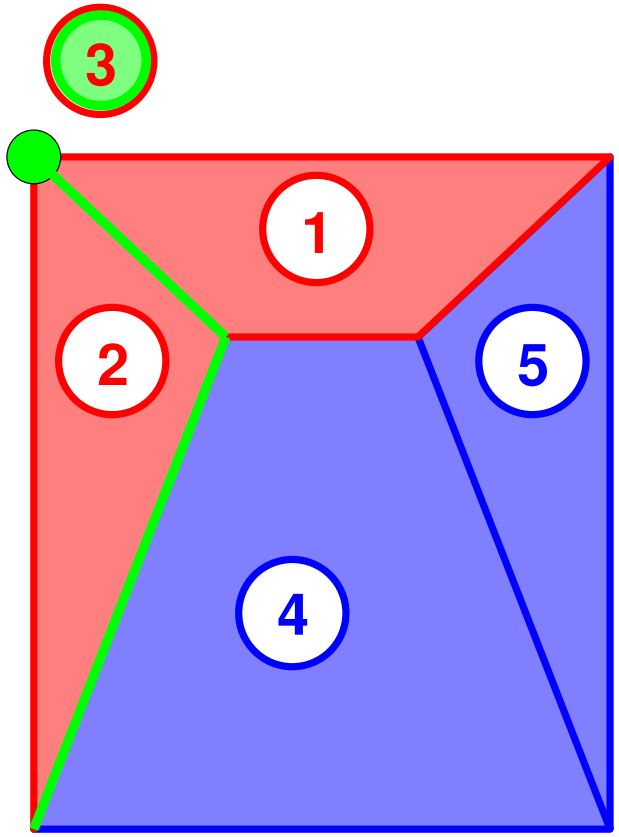

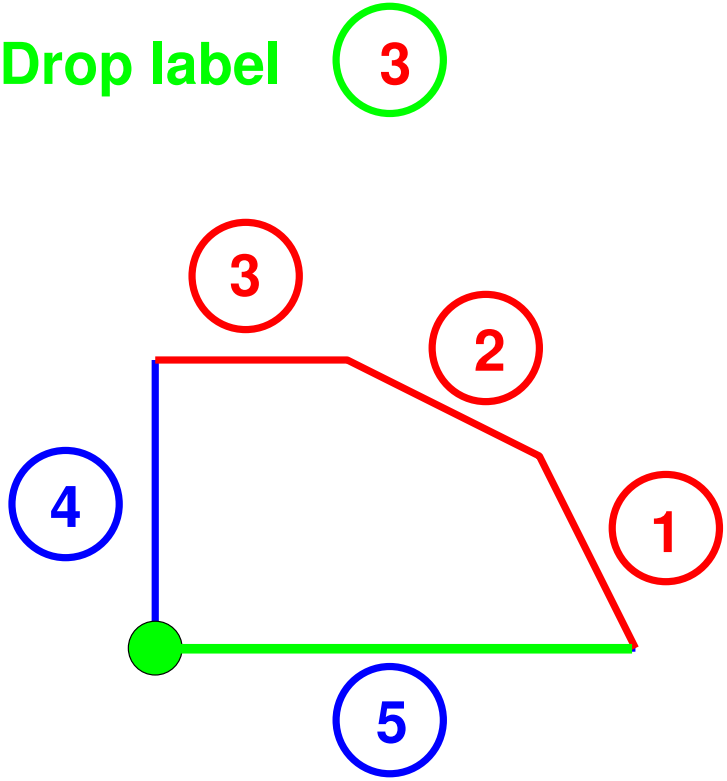

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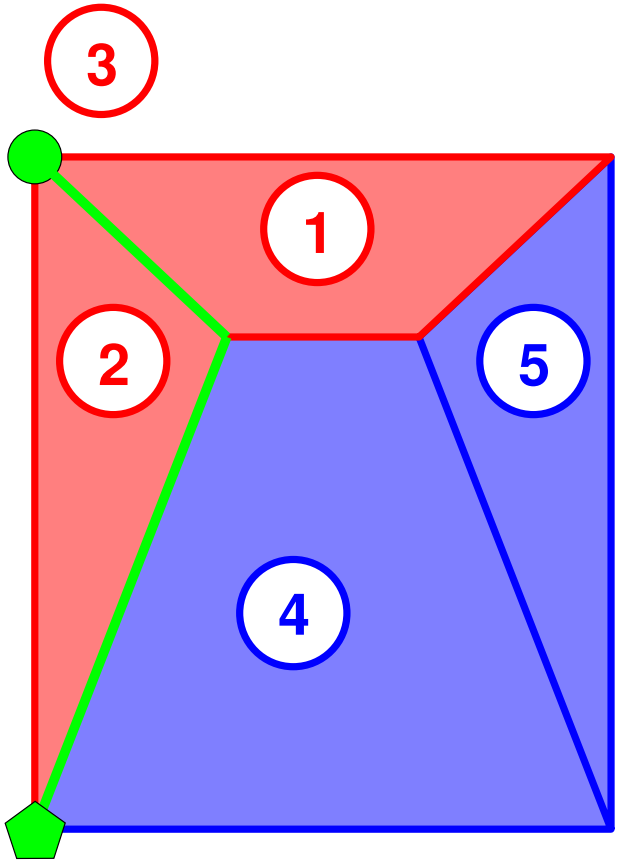
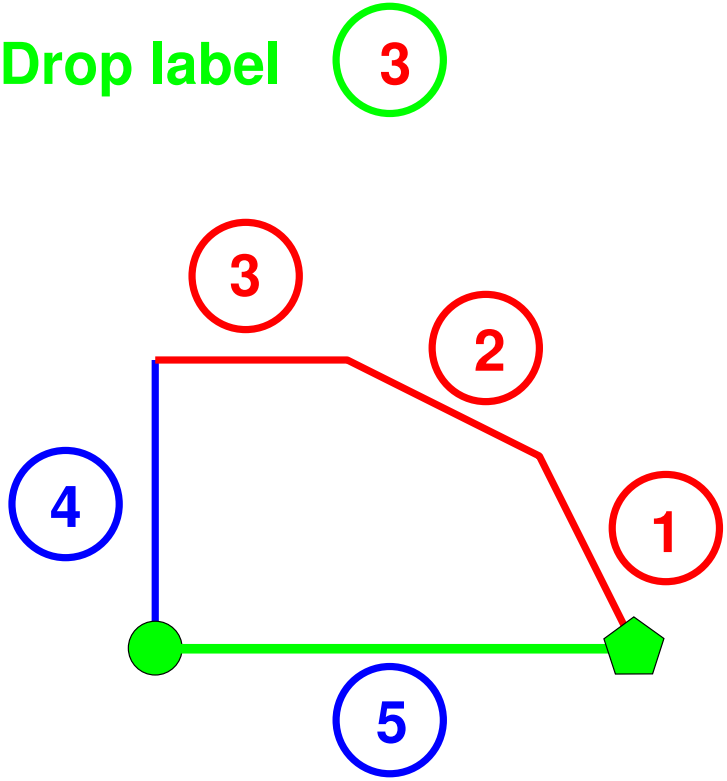




Why Lemke-Howson works

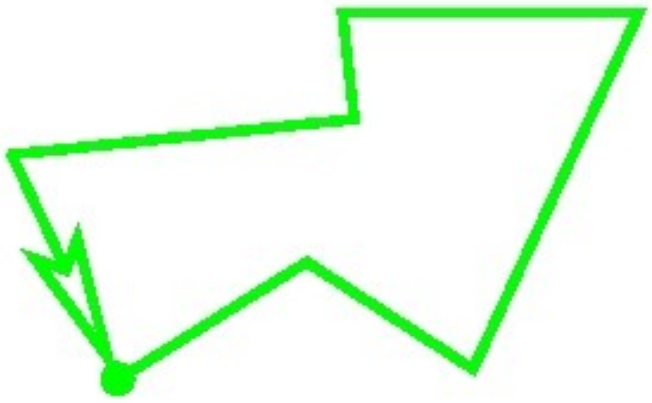
LH finds at least one Nash equilibrium because

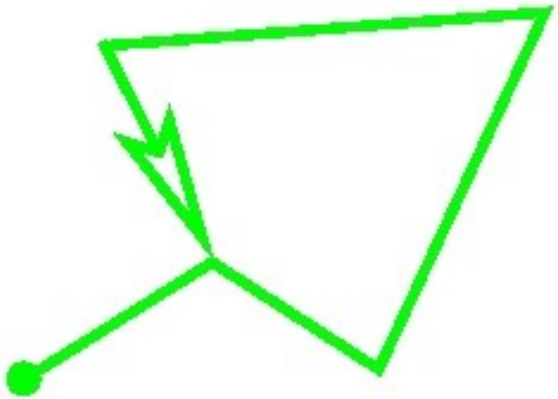
- finitely many "vertices"

for nondegenerate (generic) games:

- **unique** starting edge given missing label
- **unique** continuation

$\Rightarrow$ precludes "coming back" like here:





Complexity of Lemke-Howson

- finds at least one Nash equilibrium,
  pivots like Simplex algorithm for linear programming

- Simplex may be exponential [Klee-Minty cubes]

- exponentially many steps of Lemke-Howson
  for **any** dropped label?

- **Yes**! This is our result.

Our result

There are $d \times d$ games with exactly one Nash equilibrium, for which the Lemke-Howson algorithm takes $\geq \phi^{3d/4}$ many steps for **any dropped label** (with Golden Ratio $\phi = (\sqrt{5} + 1) / 2 = 1.618...$)

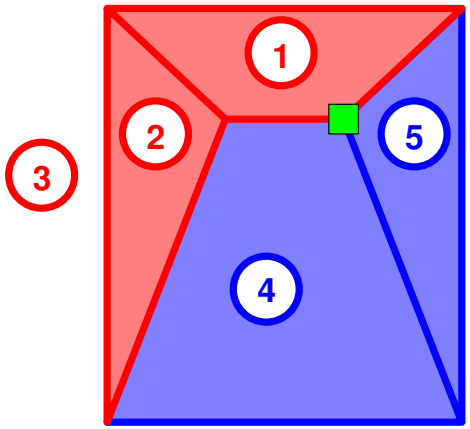
We will show this extending

[Morris 1994] - exponentially long Lemke paths (finds symmetric equilibria of symmetric games)

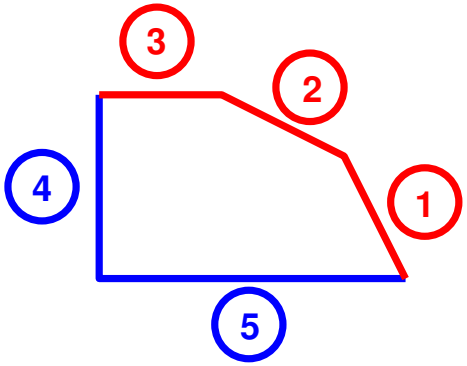
[von Stengel 1999] - games with many equilibria

using **dual cyclic polytopes**

Vertices as bit patterns



P

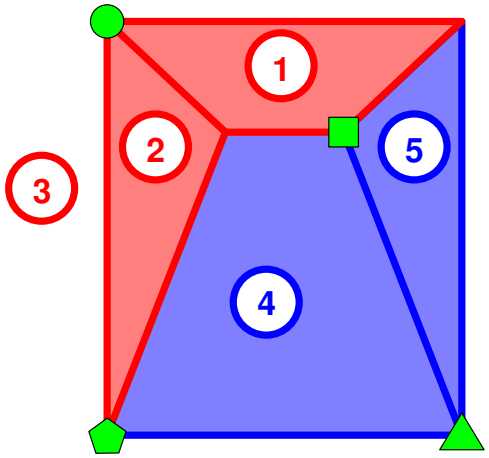


Q

1	2	3	4	5
1	1	1	0	0
1	1	0	1	0
1	0	1	0	1
1	0	0	1	1
0	0	1	1	1
0	1	1	1	0

1	2	3	4	5
0	0	0	1	1
0	0	1	1	0
0	1	1	0	0
1	1	0	0	0
1	0	0	0	1

Vertices as bit patterns



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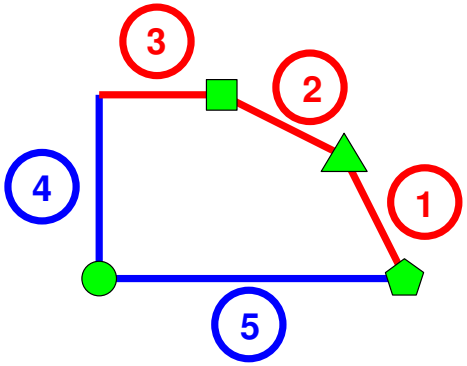


Q

	1	2	3	4	5
●	1	1	1	0	0
	1	1	0	1	0
	1	0	1	0	1
■	1	0	0	1	1
▲	0	0	1	1	1
⬟	0	1	1	1	0

1	2	3	4	5
0	0	0	1	1
0	0	1	1	0
0	1	1	0	0
1	1	0	0	0
1	0	0	0	1

Dual cyclic polytopes

- vertices = strings of **N** bits with **d** bits "**1**",
- **no** odd substrings 010, 01110, 0111110, . . .
  [Gale **even**ness]

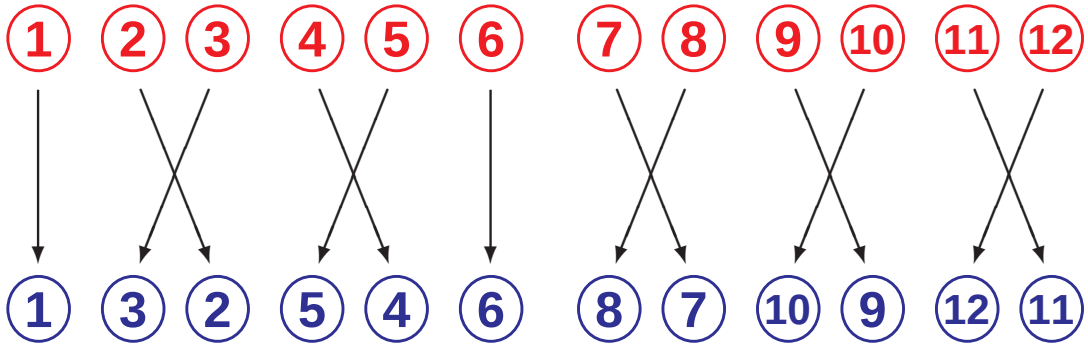
Example: d=4, N=6 d=2, N=6 (4×2 game)

d=4, N=6	d=2, N=6
111100	000011
111001	000110
110110	001100
110011	011000
101101	110000
100111	100001
011110	
011011	
001111	

Permuted labels

P = dual cyclic polytope in dimension **d** with **2d** facets

with facets labeled



Q = P

with facets labeled

$\Longrightarrow$ only **one** non-artificial equilibrium:

0 0 0 0 0 0 1 1 1 1 1 1

1 1 1 1 1 1 0 0 0 0 0 0

Lemke–Howson will take long to find it!

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1

Lemke-Howson on dual cyclic polytopes

P								Q							
1	2	3	4	5	6	7	8	1	3	2	4	6	5	8	7
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0								

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1

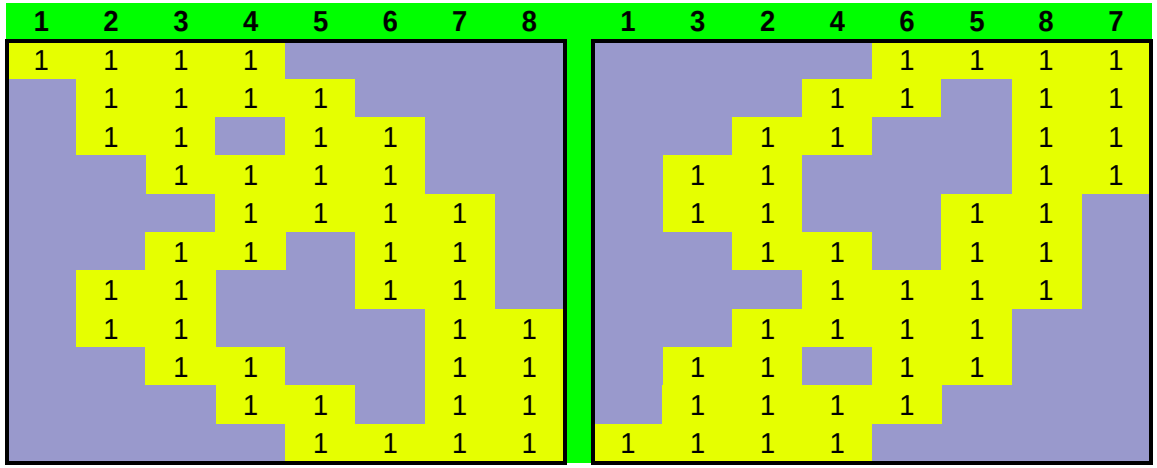
Lemke-Howson on dual cyclic polytopes

P								Q							
1	2	3	4	5	6	7	8	1	3	2	4	6	5	8	7
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0								

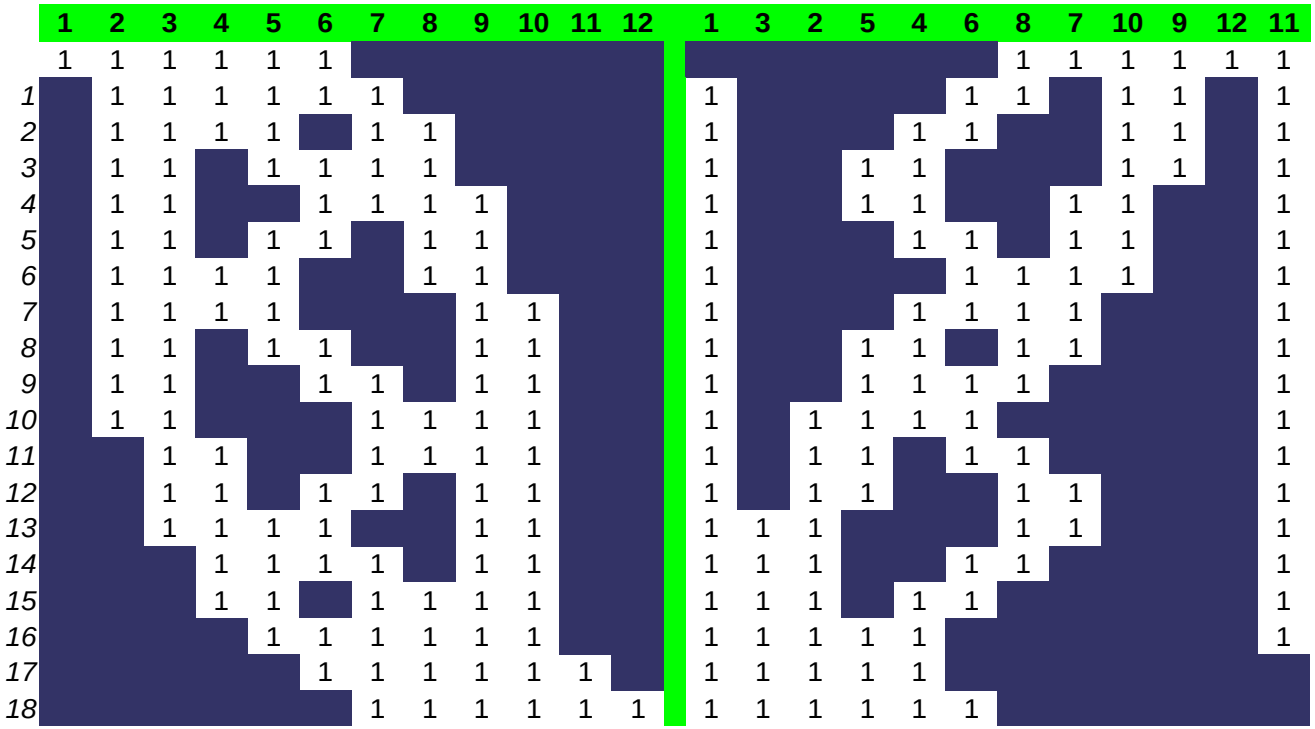
Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	0	0	1	1	0	0	1	1
0	0	1	1	1	1	0	0	0	1	1	0	0	0	1	1
0	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0
0	0	1	1	0	1	1	0	0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0
0	1	1	0	0	0	1	1	0	0	1	1	1	1	0	0
0	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0
0	0	0	1	1	0	1	1	0	1	1	1	1	0	0	0
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0

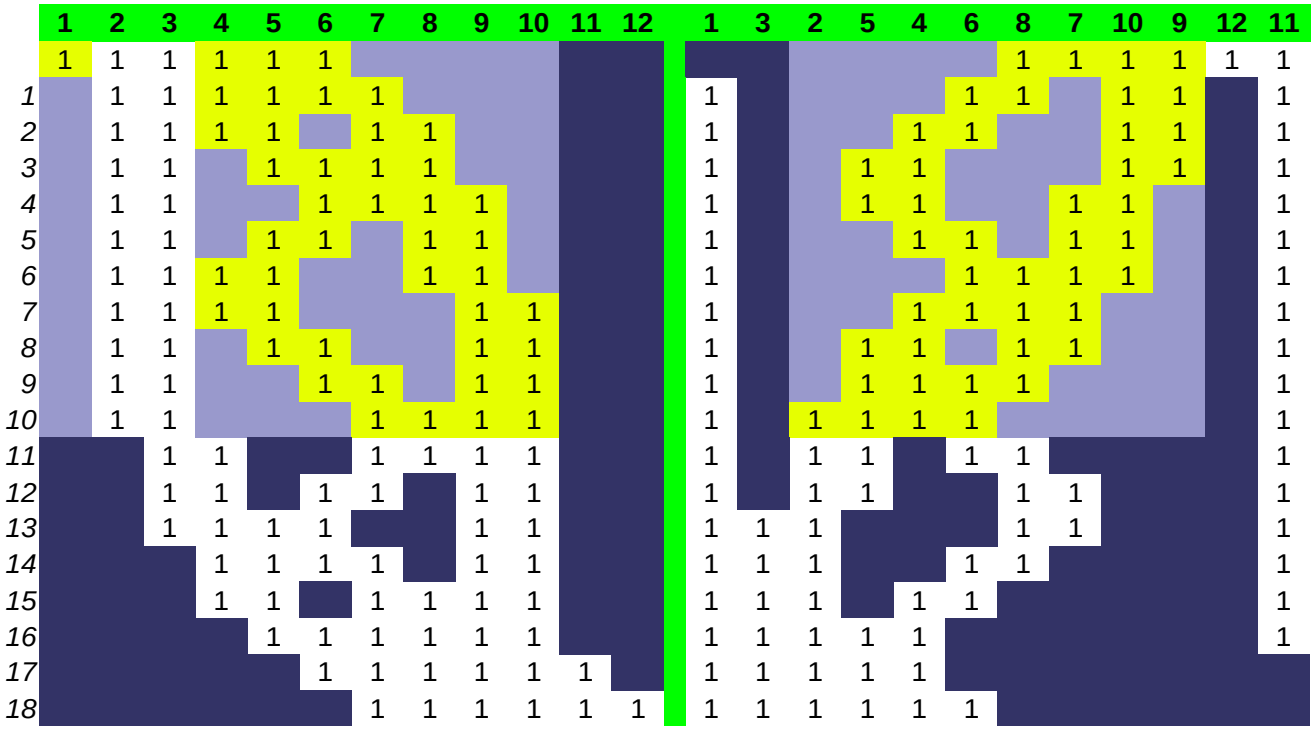
A(4) = path for d=4, label 1



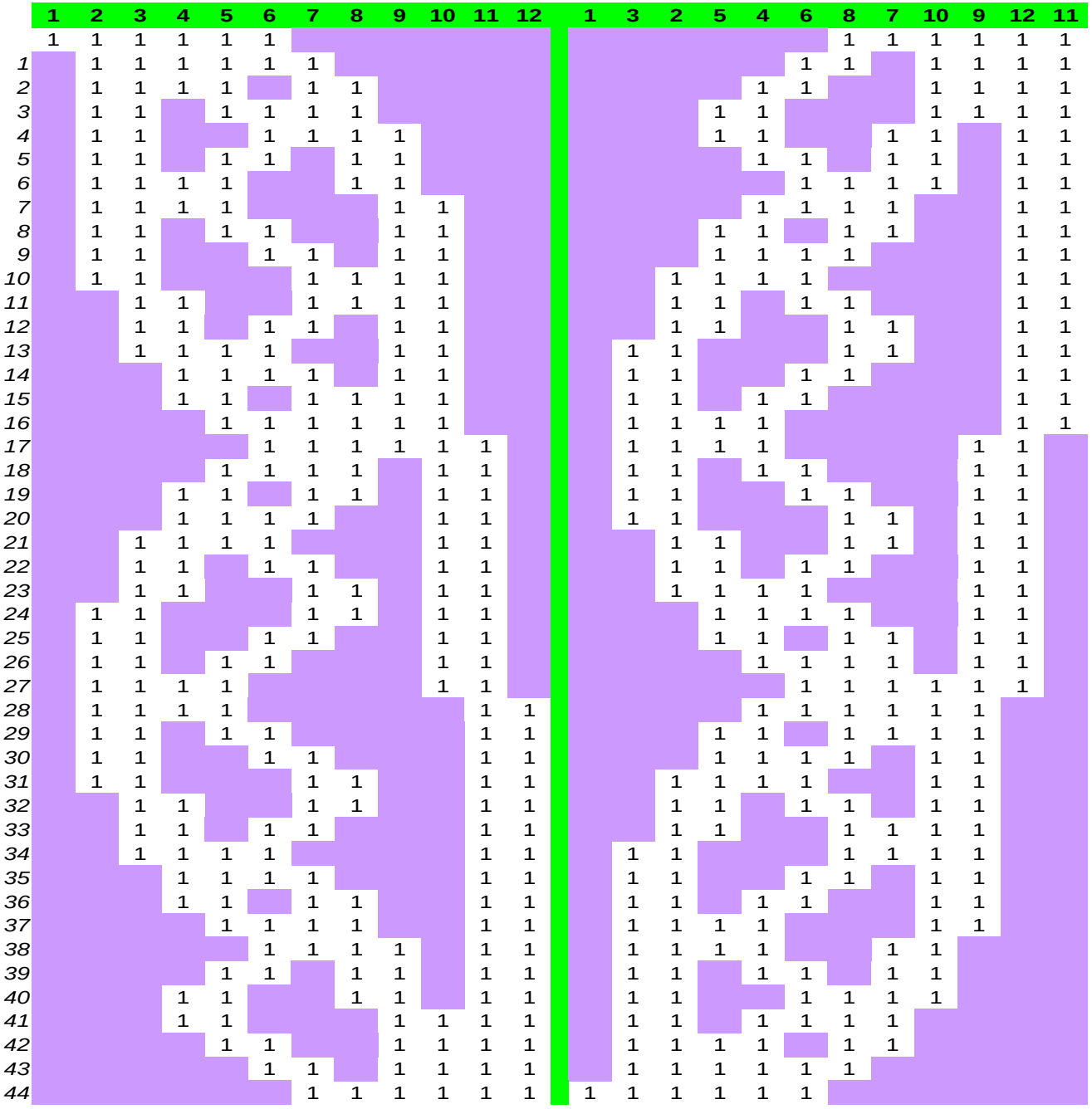
B(6) = path for d=6, label 12



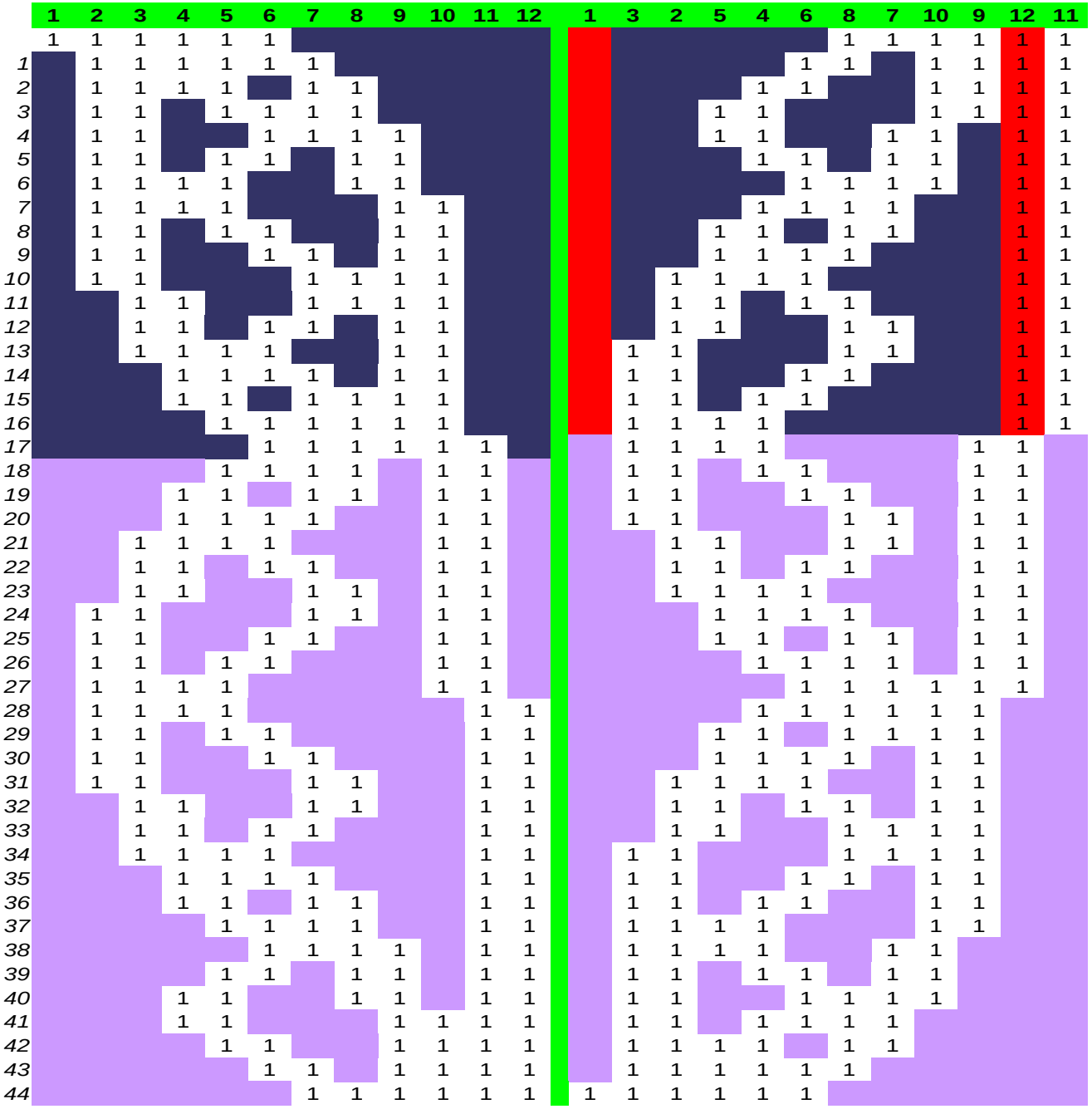
A(4) is prefix of B(6)



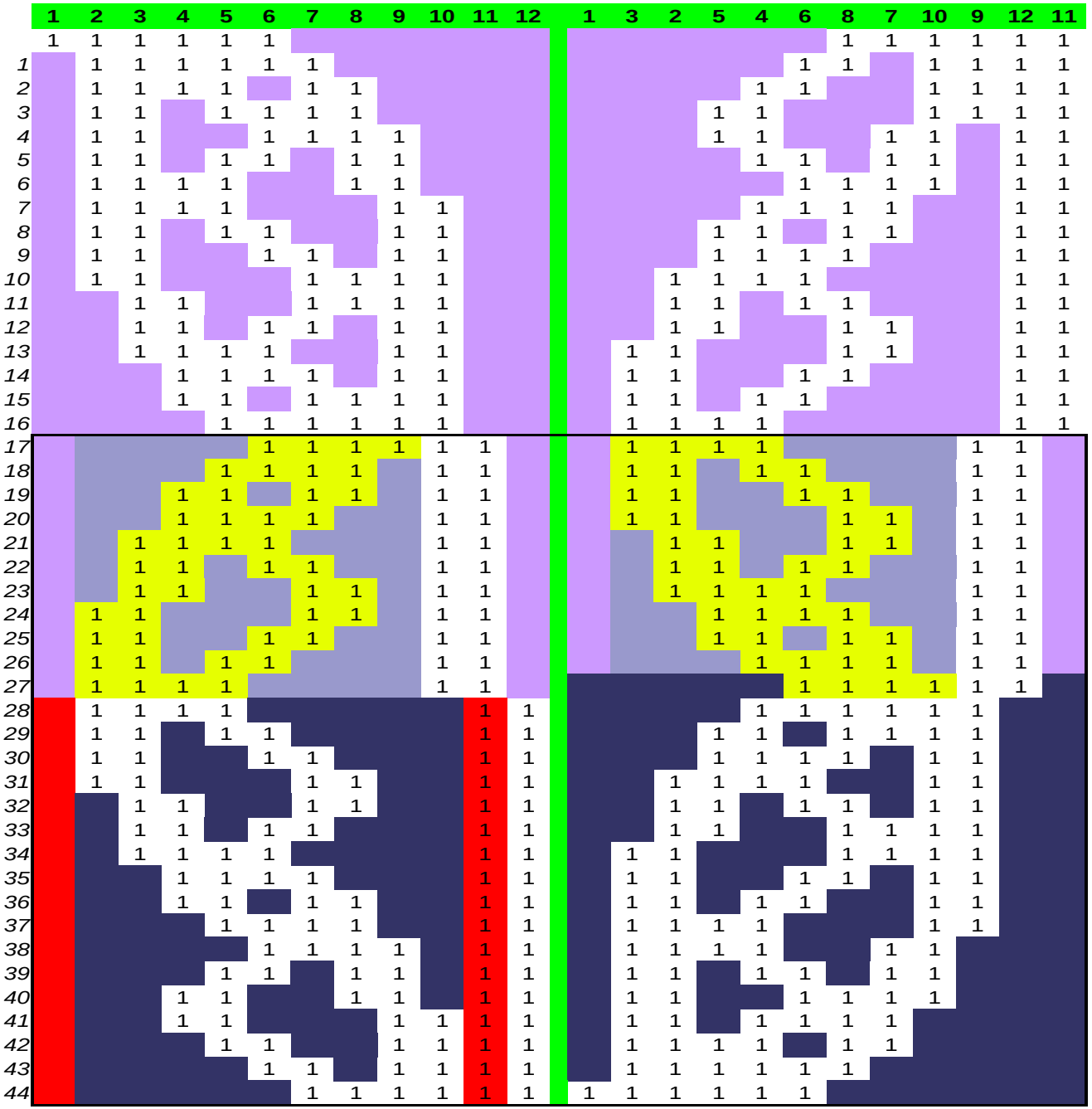
A(6) = path for d=6, label 1



B(6) is prefix of A(6)



Suffix of A(6) = C(6) = A(4)+B(6)



Recurrences for longest paths

A(d) = LH path dropping label 1 in dim d

B(d) = LH path dropping label 2d in dim d

C(d) = suffix of A(d)

lengths of

B(2) C(2) **A**(2) B(4) C(4) **A**(4) B(6) C(6) **A**(6) . . .

are the **Fibonacci** numbers

2 3 5 8 13 21 34 55 89 . . .

Exponential path lengths

longest paths: drop label 1 or 2d, paths A(d), B(d)

path length $\Omega(\phi^{3d/2})$

with Golden Ratio $\phi = (\sqrt{5} + 1) / 2 = 1.618...$

shortest paths: drop label 3d/2, path B(d/2)+B(d/2+1)

path length $\Omega(\phi^{3d/4}) = \Omega(1.434...^{d})$

So far:

- NE of a bimatrix game = combinatorial polytope problem
- label dual cyclic polytopes,
  equilibrium at end of **exponentially long** paths
- **but**: fully mixed equilibrium easily guessed
  by support enumeration algorithms

Extension to nonsquare games

- Extend to $d \times 2d$ games with hard-to-guess support (exponentially many guesses on average) **and** exponentially long paths

- Use dual cyclic polytopes with similar labeling as for $d \times d$ games.

Nonsquare games

P = dual cyclic polytope in dimension d with 3d facets

Q = dual cyclic polytope in dimension 2d with 3d facets

P's facets labeled 1 2 3 4 5 6 7 8 9 10 11 12

Q's facets labeled 1 3 2 4 6 5 8 7 10 9 12 11

equilibria:

0 0 0 0 1 1 1 1 0 0 0 0	1 1 1 1 0 0 0 0 1 1 1 1
0 0 0 0 1 1 0 0 1 1 0 0	1 1 1 1 0 0 1 1 0 0 1 1
0 0 0 0 0 1 1 0 0 1 1 0	1 1 1 1 0 1 1 0 0 1 1 0
0 0 0 0 0 0 1 1 0 1 1 0	1 1 1 1 1 1 0 0 0 1 1 0
. . .	

Equilibrium supports = exponentially small fraction of all supports