Comparing parallel and sequential Selfish Routing in the Atomic Players setting

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Outline

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Introduction

Fundamental problem :
Efficient routing in large traffic and communication networks.

Problems:

- Many independent agents.
- Load dependant latency.

Solution: Selfish routing

Selfish routing

- Each user chooses a strategy to minimize his own cost, given the other players' strategies.
- Expect the routes chosen by users to form a Nash equilibrium
- Nash equilibrium : a strategies profile such that no user has incentive to change unilaterally his own strategy.

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- Equilibria of noncooperative game are typically inefficient.

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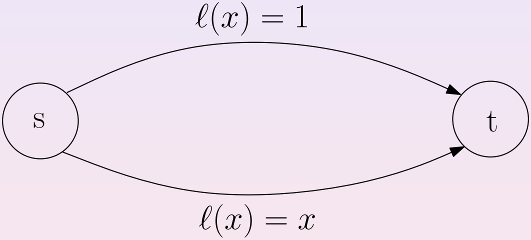
How efficient is the Selfish Routing? (Cont.)

Example : Route a one unit flow from *s* to *t*



Question : what will selfish network users do?

- assume that every user wants to have a smallest-possible delay.

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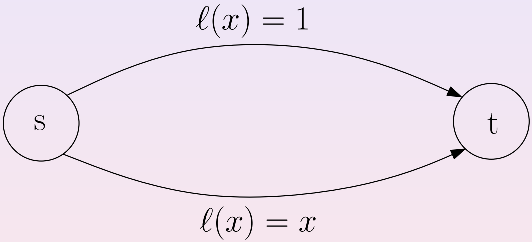

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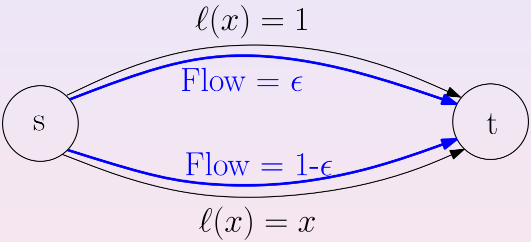

Because :

- If $\epsilon > 0$, the delay experienced by the flow on the bottom link is < 1.
- If $\epsilon = 0$, no one has incentive to move.
- All flows experience a delay of 1.

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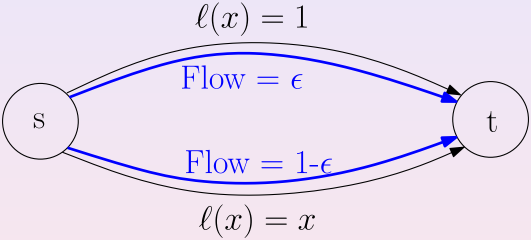

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Solution : the flow is splitted equally.

Delay :

- The top half has 1 unit of delay.
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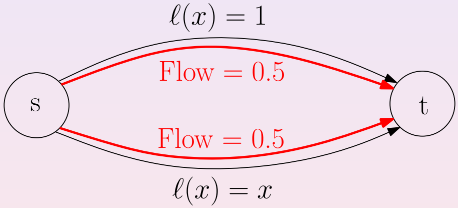

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The Price of Anarchy

Price of anarchy [Papadimitriou 2001] – “competitive analysis for noncooperative games”

Definition :

$$\text{POA} = \frac{\text{Worst case Nash equilibrium}}{\text{Optimal solution}}$$

Price of anarchy measures the “price” of not having central coordination in system.

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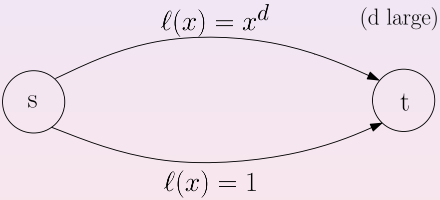

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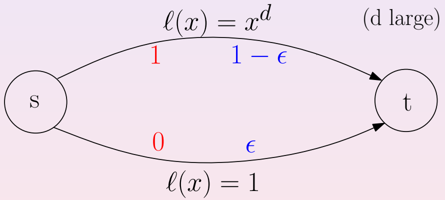

Social cost :

- Nash: $1 \cdot 1^d + 0 \cdot 1 = 1$
- Optimal: $\epsilon \cdot 1 + (1 - \epsilon) \cdot \epsilon^d \longrightarrow 0$ when ϵ close to 0

General latency function(cont.)

Different approach :

Bicriteria Results[Roughgarden/Tardos 2000] : for every network,

the cost of Nash flow of traffic rate $r \leq$ the cost of minimum flow of traffic rate $2r$.

Linear latency function

Theorem [Roughgarden/Tardos 2000] :

If latency function is of the form : $\ell_e(x) = a_e x + b_e$ where a_e and $b_e \geq 0$,

the cost of a Nash flow is at most 4/3 times that of the minimum-latency flow.

Traffic Equilibrium Paradoxes

Intuitively, less players $\Longrightarrow$ smaller social cost

- does not hold in general
- Numerical example[Catoni/Pallottino 1991] :
- Route a flow of 630 from s_1 to t_1 and another 630 from s_2 to t_2.

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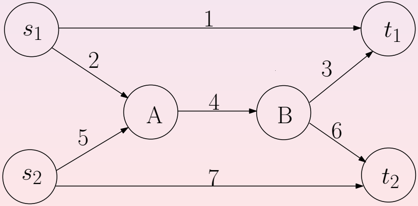

$$\ell_1 = f_1 + 30;$$
$$\ell_4 = f_4 + 60;$$
$$\ell_7 = f_7$$
$$\ell_2 = \ell_3 = \ell_5 = \ell_6 = 0$$

Numerical example

- Consider four models :
  - **User Equilibrium** – a large population of very small network users
  - **System Equilibrium** – a single user
  - **Mixed Behavior Equilibrium** – one CN-player with many very small network users
    $\Longrightarrow$ CN-player – a player controls an amount of splittable flow
  - **Cournot-Nash Equilibrium** – two CN-players
- Number of players : $UE > ME > CNE > SE$

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Traffic Equilibrium Paradoxes(Cont.)

	UE	ME	CNE	SE
$f(1,1)$	420	360	382	420
$f(1,4)$	210	270	248	210
$f(2,4)$	180	150	238	195
$f(2,7)$	450	480	392	435
$C(1)$	283500	270000	292792	286650
$C(2)$	283500	302400	283612	279900
Social cost	**567000**	**572400**	**576404**	**566550**

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Mathematical Model

- A single commodity network congestion game.
- 2 parallel edges.
- m CN-players, each controls one unit of splittable flow.
- A linear latency function $\ell_e(x) = a_e x + b_e$.
- Cost experienced by player i : $c_i = \sum_{e \in E} \ell_e(f_e) \cdot f(i, e)$.

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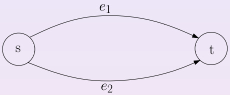
- Players make decision simultaneously.
- Expect players to form a Nash equilibrium.



Property 1 : At Nash equilibrium, every player splits his flow in the same proportion.

Property 2 : If latency functions are equal, every player splits his flow equally among edges.

Property 3 : At Nash equilibrium, every player incurs the same cost.

- All properties hold for any number of links

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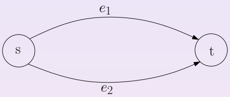

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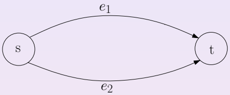

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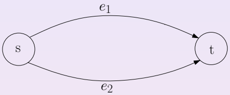

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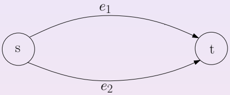
- Perfect information – user is perfectly informed of all the event that have previously occurred.



- We solve the game using backward induction.
  - stage m : player m solves $\min c_m(a_1, a_2, \ldots, a_m)$ given the actions $a_1, \ldots, a_{m-1}$ previously chosen. Denote the solution by $R_m(a_1, \ldots, a_{m-1})$.
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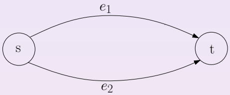

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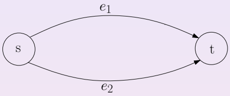

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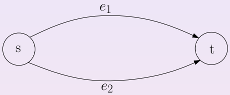

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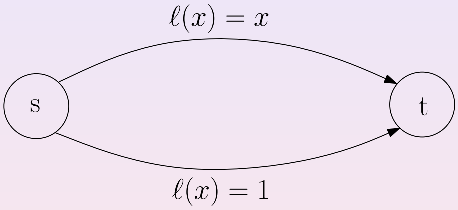

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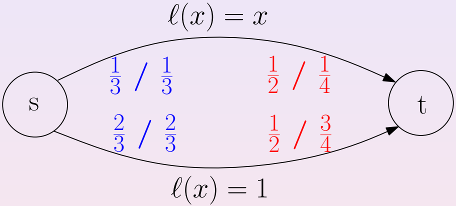

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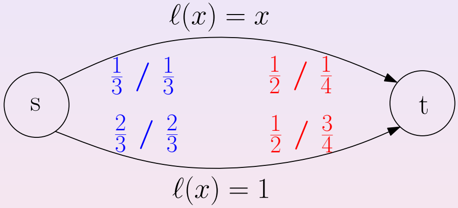

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Some results

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For 2 players,

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