A Scan Markov Chain for Sampling Colourings

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Outline

Proper Colouring of Graphs

Computational problem

Want to sample efficiently from the (nearly) uniform distribution of proper q-colourings of a graph with maximum vertex degree Δ using a systematic approach.

Definition

A proper colouring of a graph is an assignment of a colour to each site where no two adjacent sites have the same colour.

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Ω is the set of all proper colourings.

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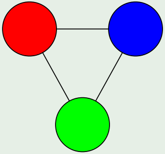
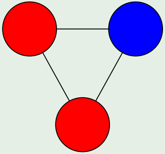
Example

A proper colouring



An improper colouring



Markov Chains and Sampling

A Markov chain is a random process $X_0, X_1, \ldots$

- each state X_k takes a value in Ω (state space), and
- the transition at any time depends only on the current state

If q is sufficiently large then a Markov chain converges to its stationary distribution (subject to some conditions!)

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Scan Markov Chains

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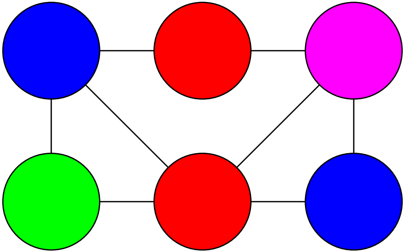
A Markov chain on Ω is called a scan if the sites are updated in a deterministic order.

- An update is a randomised procedure. E.g. heat bath.



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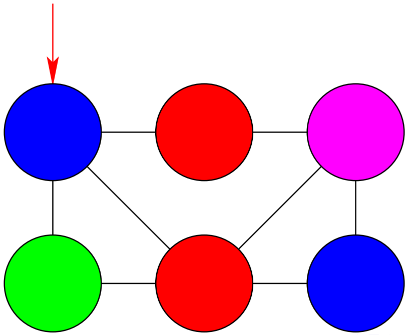

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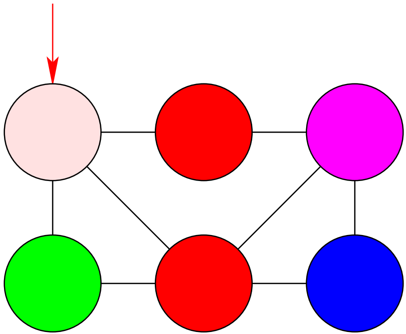

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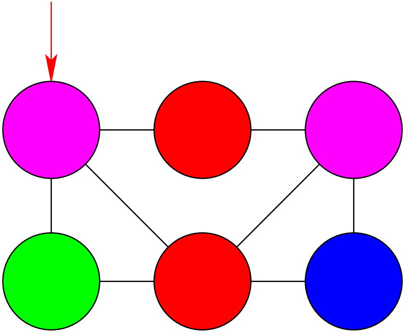

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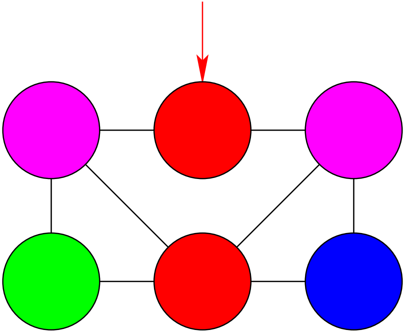

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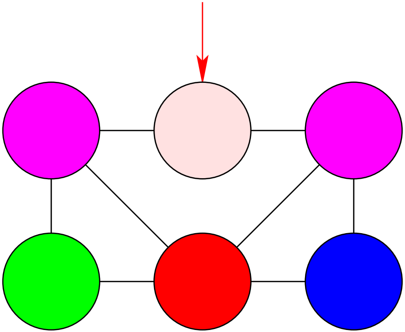

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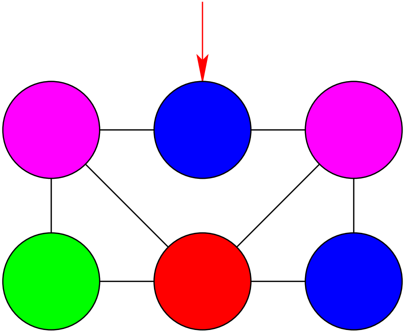

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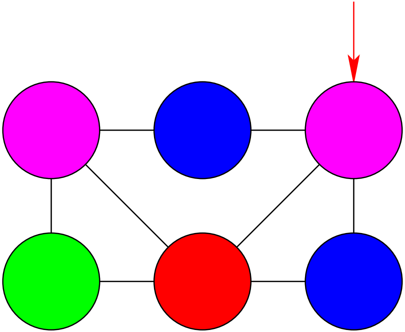

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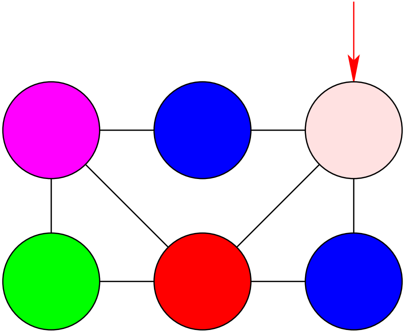

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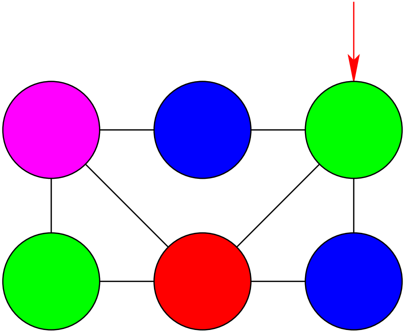

Mixing Time of Markov Chains

Definition

- The mixing time of a Markov chain is how long it takes to become sufficiently close to its stationary distribution.
- A Markov chain is rapidly mixing if the mixing time is at most polynomial in the size of the graph.

Computational question

Given a systematic scan $\mathcal{M}$

- For what values of q (in terms of Δ) is $\mathcal{M}$ rapidly mixing?

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Previous Work

Systematic scan:

- $O(n \log n)$ mixing, $q > 2\Delta$. General graphs (Dyer, Goldberg and Jerrum, 2005)
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Random update:

- $O(n \log n)$ mixing, $q > \frac{11}{6}\Delta$. General graphs (Vigoda, 2000)

Overview of Results

General Mixing Result

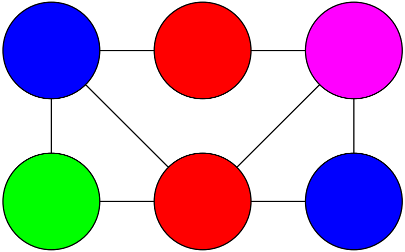
We show a condition for $O(n \log n)$ mixing of an arbitrary systematic scan.

Application

- $O(n \log n)$ mixing when $q \geq 2\Delta$ on general graphs

Block Moves

- A block move considers a set of sites for simultaneous update.
- Scan: set of blocks Θ must cover V. A systematic scan updates the blocks in a deterministic order.



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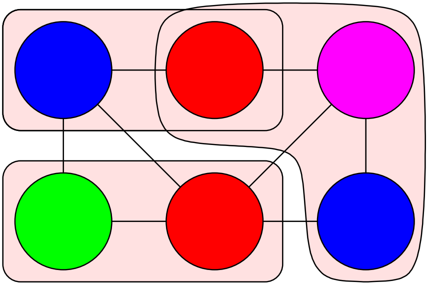

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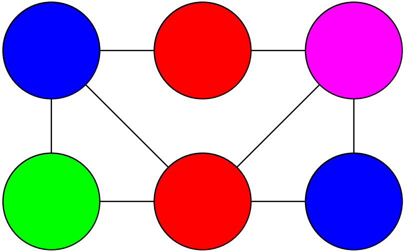

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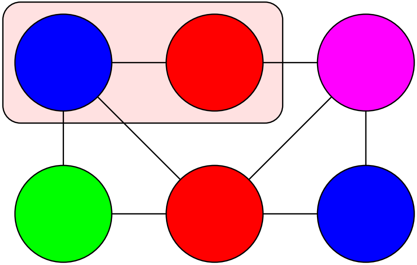

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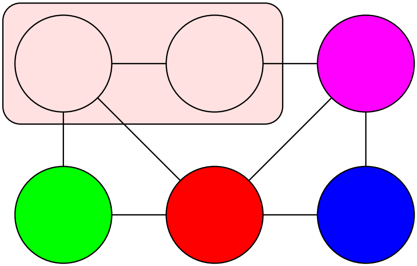

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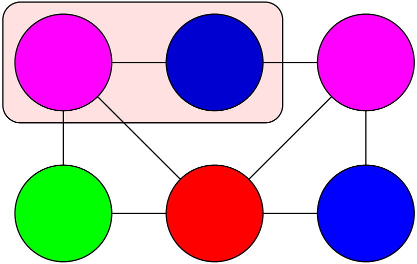

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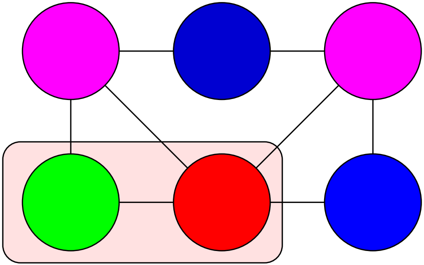

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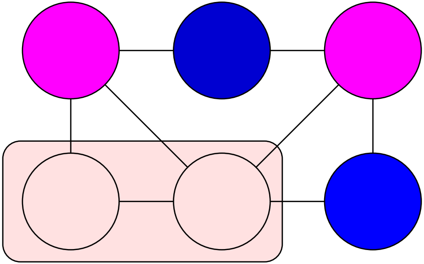

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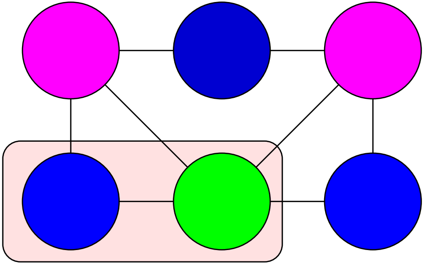

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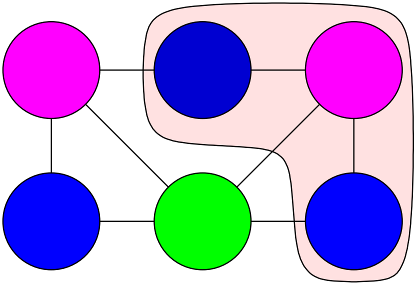

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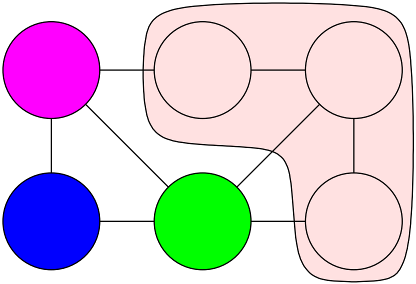

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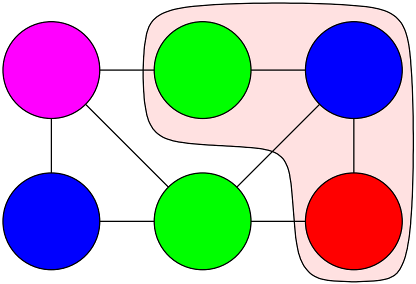

Coupling

Definition

A coupling of two distributions X and Y on Ω is a distribution ψ on $\Omega \times \Omega$ such that when (x, y) is drawn from ψ then the distribution of x is X and the distribution of y is Y.

Example

Let $\Omega = \{S_1, S_2\}$
X, Y distributions on Ω.

- $\mathbf{Pr}_X(S_1) = 1/2$ and
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$\mathbf{Pr}_\psi$	X	Y
$1/3$	S_1	S_1
$1/2$	S_2	S_2
$1/6$	S_1	S_2

Influence *on* a site

Consider two copies X and Y of the same M.C

- X and Y differ only at the colour of site i
- block Θ_k is going to be updated
- $P^{[k]}$ is the transition matrix for updating block Θ_k

Definition

The influence site i has on site j is the maximum probability that the two copies of the M.C will differ at site j in some coupling of the distributions $P^{[k]}(X, \cdot)$ and $P^{[k]}(Y, \cdot)$. This is denoted by $\rho^k_{i,j}$.

Total Influence *on* a site

- Total influence on site j (under block Θ_k) is

$$\sum_{i \in V} \rho_{i,j}^{k}$$

Definition

The maximum influence on any site is

$$\alpha = \max_{k \in \Theta} \max_{j \in V} \sum_{i \in V} \rho_{i,j}^{k}$$

Mixing Time of Systematic Scan

Definition

$$\alpha = \max_{k \in \Theta} \max_{j \in V} \sum_{i \in V} \rho^k_{i,j}$$

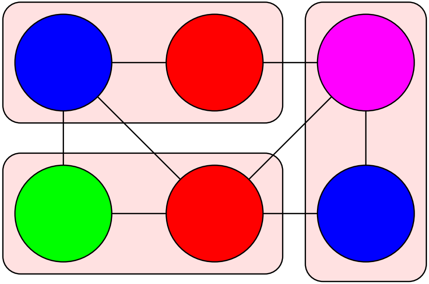
Theorem

$\mathcal{M}$ is any systematic scan with block set Θ. If $\alpha < 1$ then

$$\mathrm{Mix}(\mathcal{M}, \epsilon) \leq O\left(\frac{n \log(n\epsilon^{-1})}{1 - \alpha}\right)$$

Application: Edge Scan

- $G = (V, E)$ is any graph
- $\Theta \subseteq E$
- Update rule is heat bath



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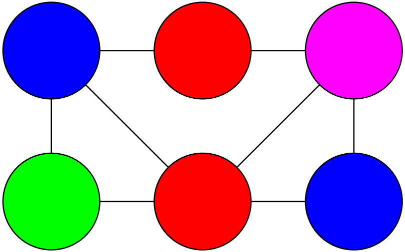

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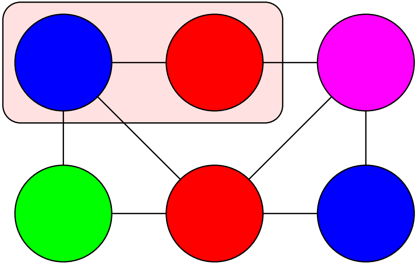

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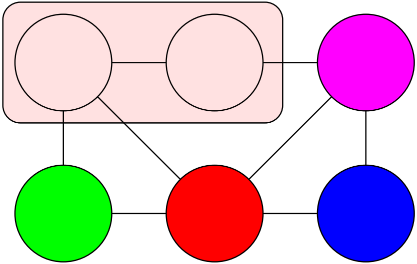

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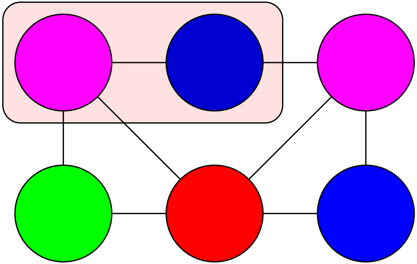

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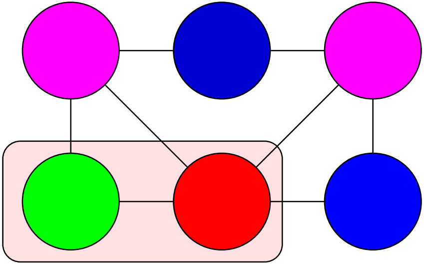

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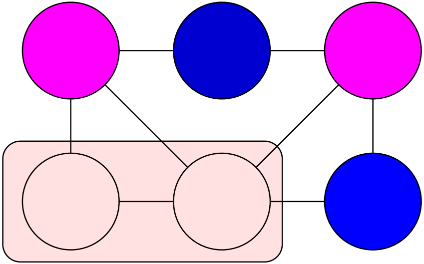

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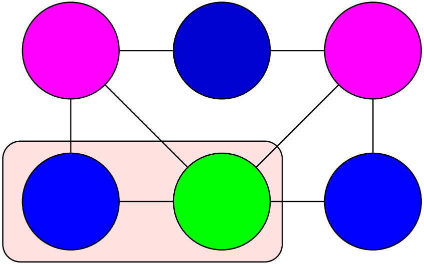

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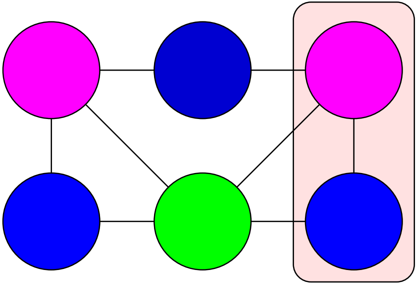

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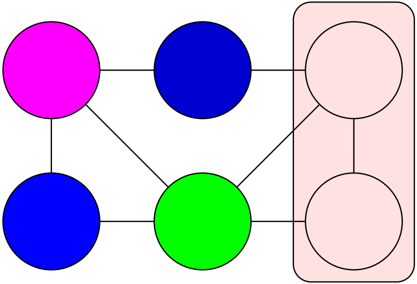

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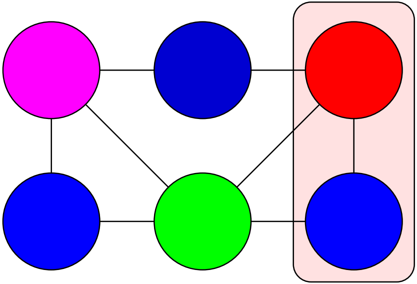

Mixing Time of Edge Scan

Case analysis gives

$$\alpha < 1 - \frac{1}{\Delta^2}$$

when $q \geq 2\Delta$ so

$$\mathrm{Mix}(\mathcal{M}_{-}, \epsilon) \leq \Delta^2 n \log\left(n\epsilon^{-1}\right)$$

Remark

This is the first general $O(n \log n)$ mixing result for scan when $q = 2\Delta$.

Summary

- Showed a condition for $O(n \log n)$ mixing of systematic scan
- Used the condition to prove $O(n \log n)$ mixing for general graphs provided $q \geq 2\Delta$.