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Containing Families

An introduction to Containers

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A Problem. . .

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- Find a theory of datatypes that supports reusable libraries of code
  - Boilerplate
  - Generics — Haskell deriving

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- Vectors:

$$\underline{\text{data}}\ \frac{n : \mathsf{Nat}\ ;\ A : \star}{\mathsf{Vec}\ n\ A : \star}\ \underline{\text{where}}\ \frac{}{\mathsf{nil} : \mathsf{Vec}\ 0\ A}\ ;\ \frac{a : A\ ;\ as : \mathsf{Vec}\ n\ A}{\mathsf{cons}\ a\ as : \mathsf{Vec}\ (1{+}\ n)\ A}$$

Positivity (Luo)

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- In the current version we use Luo's syntactic definition of strictly positive inductive families. This doesn't allow us to have:

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- Why would we want this anyway?
- The type we are defining can only appear to the right of arrows:

$$\frac{f : \mathsf{Bool} \to \mathsf{X}}{\mathsf{c}\ f : \mathsf{X}}$$

But hang on. . .

- What about definitions like the following:

$$\frac{ts : \mathsf{List}\ (\mathsf{RoseTree}\ A)}{\mathsf{node}\ ts : \mathsf{RoseTree}\ A}$$

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- We can have an encoding of Rose Trees, but they won't be the ones we want!
  - The definition above is the most concise and intuitive
  - We want to use the real List functions to map across sub-nodes

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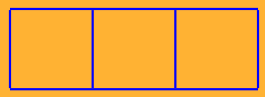
Lists have a shape very similar to the natural numbers. . .



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Containers have a set of Shapes. $S : \star \ldots$
and for each shape, some set of Positions where data goes
$P : \forall s : S \Rightarrow \star$

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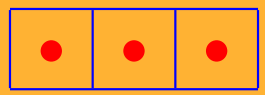

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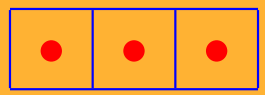

Trees: The shape of Binary trees with data at the leaves is a Binary Tree with no room for data, Positions are then paths to leaves...

More Containers

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- Or define codes for types ($\mu, +, \times \ldots$) and for each code a type of elements
  - Generic Equality, Map, Differentiation...
  - See previous BCTCS talk

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$$\frac{I, O : \star}{\mathsf{IC}\ I\ O : \star}\ \underline{\text{where}}\ \frac{\begin{array}{c} S : O \to \star \\ P : \forall o : O; s : S\ o \Rightarrow I \to \star \end{array}}{S \triangleleft P : \mathsf{IC}\ I\ O}$$

Roses again

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- We can compose containers
  Given a container, with input index type J and output O, and a sub-container, with inputs I and outputs J, their composition is indexed by I and O.

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- More generally, we have a compositonal *semantic* notion of strict positivity for datatypes.

ICs and Generics

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- If we write programs parametrised by these codes and interpretations we have generic programs.
  - Generic Equality, Map, Modalities?, Differentiation
  - Composition + generics = reusable libraries for Epigram

SPF

$$\underline{\text{data}}\ \frac{\vec{I} : \mathsf{Vec}\ \star\ n \quad O : \star}{\mathsf{SPF}\ \vec{I}\ O : \star}$$

$$\underline{\text{where}}\ \frac{}{\text{`Z'} : \mathsf{SPF}\ (\vec{I} :: O)\ O} \quad \frac{T : \mathsf{SPF}\ \vec{I}\ O}{\text{`wk'}\ T : \mathsf{SPF}\ (\vec{I} :: I)\ O}$$

$$\frac{f : \forall t : \mathsf{Fin}\ n \Rightarrow \mathsf{SPF}\ \vec{I}\ O}{\text{`Tag'}\ f : \mathsf{SPF}\ \vec{I}\ (O \times \mathsf{Fin}\ n)} \quad \frac{}{\text{`0'}, \text{`1'} : \mathsf{SPF}\ \vec{I}\ O}$$

$$\frac{f : O \to O' \quad T : \mathsf{SPF}\ \vec{I}\ O}{\text{`}\Sigma\text{'}O\ f\ T : \mathsf{SPF}\ \vec{I}\ O'} \quad \frac{f : O' \to O \quad T : \mathsf{SPF}\ \vec{I}\ O}{\text{`}\Delta\text{'}O\ f\ T : \mathsf{SPF}\ \vec{I}\ O'}$$

$$\frac{f : O \to O' \quad T : \mathsf{SPF}\ \vec{I}\ O}{\text{`}\Pi\text{'}O\ f\ T : \mathsf{SPF}\ \vec{I}\ O'} \quad \frac{T : \mathsf{SPF}\ (\vec{I} :: O)\ O}{\text{`}\mu\text{'}\ T : \mathsf{SPF}\ \vec{I}\ O}$$

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- Our syntax can represent all strictly positive inductive families.
- It is *itself* a strictly positive inductive family.
- We can give the code for the code, modulo universe sizes. Epigram in Epigram?
- Strong theory of containers (Abbott, Altenkirch, Ghani, Hancock, McBride) can be used to justify these constructs.

Future directions

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- SPFs are a large universe of types
  - The larger the universe, the less structure, the fewer generic programs it can support
  - Our equality already uses Regular Families (ω-continuous functors) because we need to know there are only finitely many positions
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- Work in progress
  - Connection to high level syntax
  - How to make writing generic programs as easy as possible within the language
  - Actually implement this stuff in Epigram2

The End

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- More information:
  - Containers: http://sneezy.cs.nott.ac.uk/containers
  - Epigram: http://www.e-pig.org

- Thanks for your attention!