Correlated Equilibria in Succinct Games

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Representation of Games

- **strategic form** (normal form)

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- utility matrix for pure strategies
- example: chicken game

	stop	go
stop	4,4	1,5
go	5,1	0,0

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- chicken game:

0	1
0	0

0	0
1	0

1/4	1/4
1/4	1/4

0	1/2
1/2	0

1/3	1/3
1/3	0

Succinct Games [Papadimitriou 2005]

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- chicken game: symmetric (all players with same strategies)

  ⇒ number of players for each strategy sufficient

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Proof:

- Nash equilibria are correlated. [NASH 1954]
- Min-Max Theorem of linear programming. (HART/SCHMEIDLER 1989)
- Stationary distributions of finite Markov chains. (NAU/MCCARDLE 1990) $\longrightarrow$ STOC 2005: used for polynomial-time construction of CE.

Linear Programming Duality

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- **(D) with polynomially many variables**
- **and exponentially many constraints**

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- in each iterate K: generate candidate solution y^K (infeasible)

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- but precision issues: need exact y^K

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- **round x^K with fixed precision**

- **end with polynomially many distributions** x^K

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- **chicken game: symmetric product distributions**

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- but: UX^T has only polynomially many entries

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- **still a product distribution**
- **a game is said to have the polynomial expectation property [PAPADIMITRIOU 2005] if: exists poly-time algorithm E for p's utility on strategy i under given distribution**

Polynomial Expectation Property (2)

observe: two invocations of E compute an entry of UX^T:

$$\left[E(z,p,x^K(p \leftarrow i)) - E(z,p,x^K(p \leftarrow j))\right] x^K_{is^{-p}}$$

is equivalent to

$$\sum_{s^{-p} \in S^{-p}} \left[u^p(is^{-p}) - u^p(js^{-p})\right] \cdot x^K_{is^{-p}}$$

Linear Programming: 2nd invocation

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- **FARKAS' Lemma: (P') has non-zero solution α**

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Finish with Correlated Equilibrium

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- **gives correlated equilibrium.** $\square$

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- take as input the polynomially many x^K and the non-zero α
- first choose one of the x^K
- then within this, choose a strategy for each player (according to distribution)
- this suggests one strategy for each player

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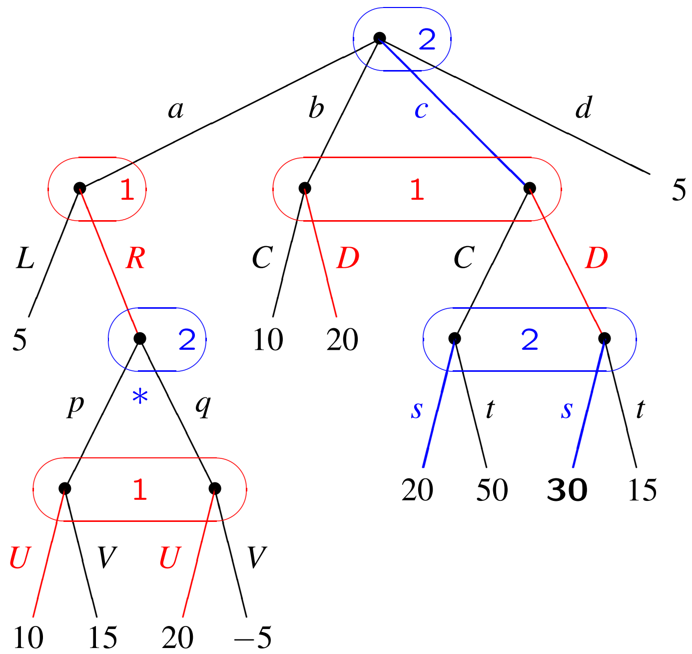
- requires specific algorithm for product distributions
- also polynomial expectation property needs adaptation

Open for extensive-form games

- **Trees with imperfect information**



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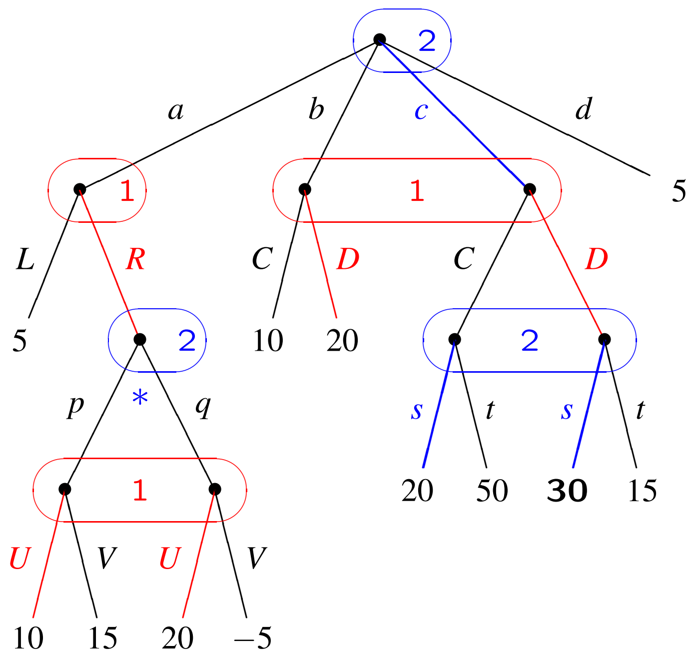

- **Number of pure strategies exponential in size of extensive game**

References

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