Improved mixing bounds for the anti-ferromagnetic Potts model on Z^2

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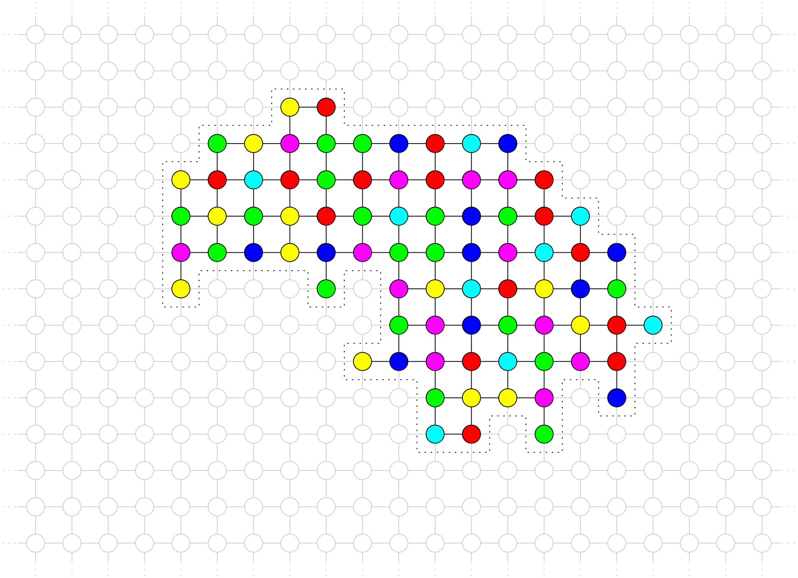
Joint work with Leslie Ann Goldberg, Russell Martin and Mike Paterson

British Colloquium for Theoretical Computer Science, 2006

Colourings

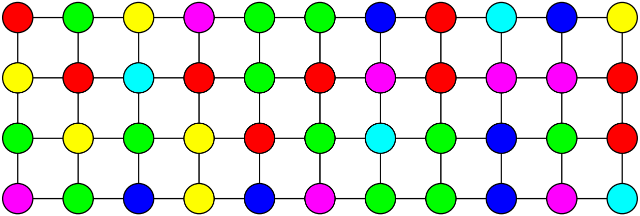


Anti-ferromagnetic Potts model

Two parameters

- q, number of colours
- $0 \leq \lambda \leq 1$

$$\text{Weight of colouring} = \lambda^{\text{number of monochromatic edges}}$$



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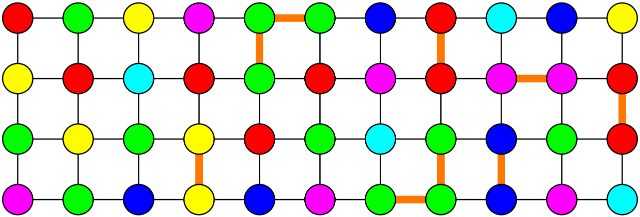


9 monochromatic edges, weight of colouring = λ^9

Distributions of colourings

$$\sigma = \text{ colouring}$$

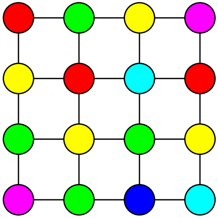
$$\text{Prob}(\sigma) = \frac{\text{weight}(\sigma)}{Z}$$

$$Z = \sum_{\text{colourings } \sigma} \text{weight}(\sigma)$$

Distributions of colourings

Uniform distribution of

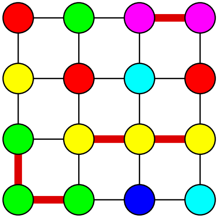
proper colourings



$\lambda = 0$

$\text{weight}(\sigma) = 1 \text{ or } 0$

all colourings



$\lambda = 1$

$\text{weight}(\sigma) = 1$

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- Want to sample from the distribution of colourings.

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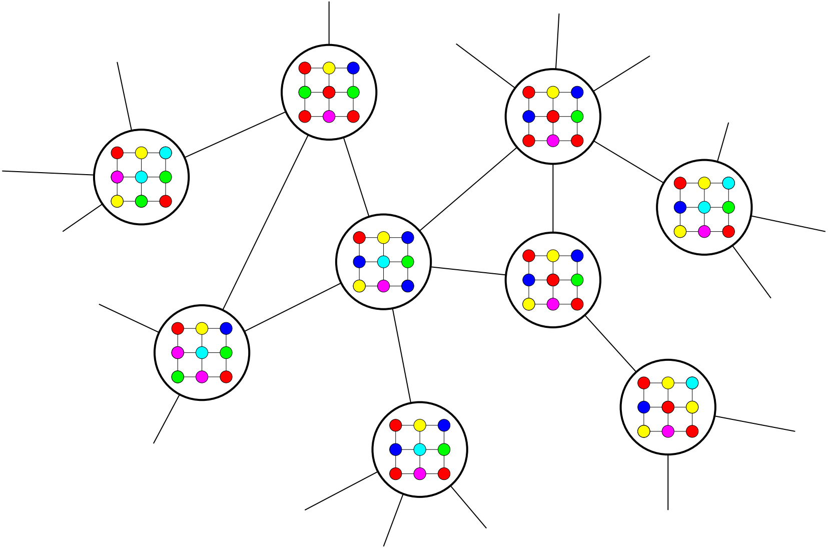
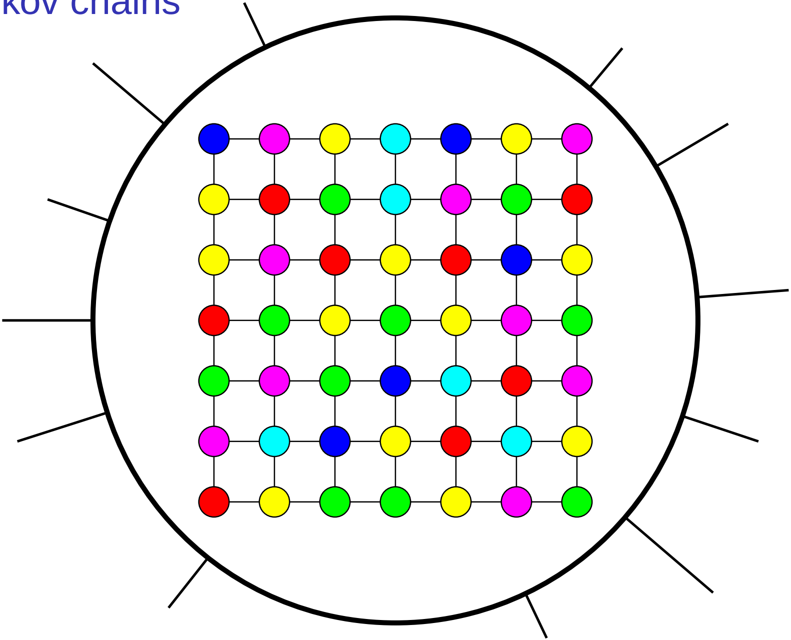
- Efficiently means polynomial time, in size of the region.

Method: Markov chains

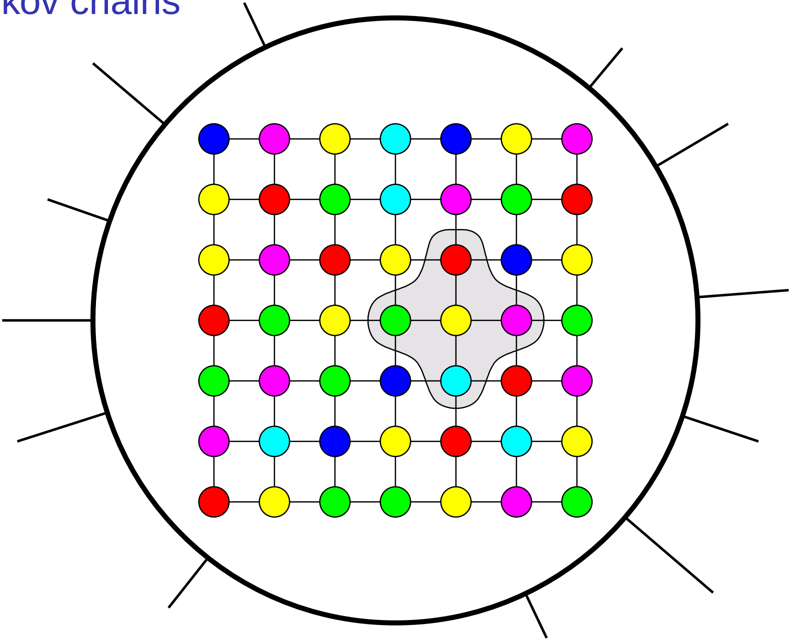
Markov chains



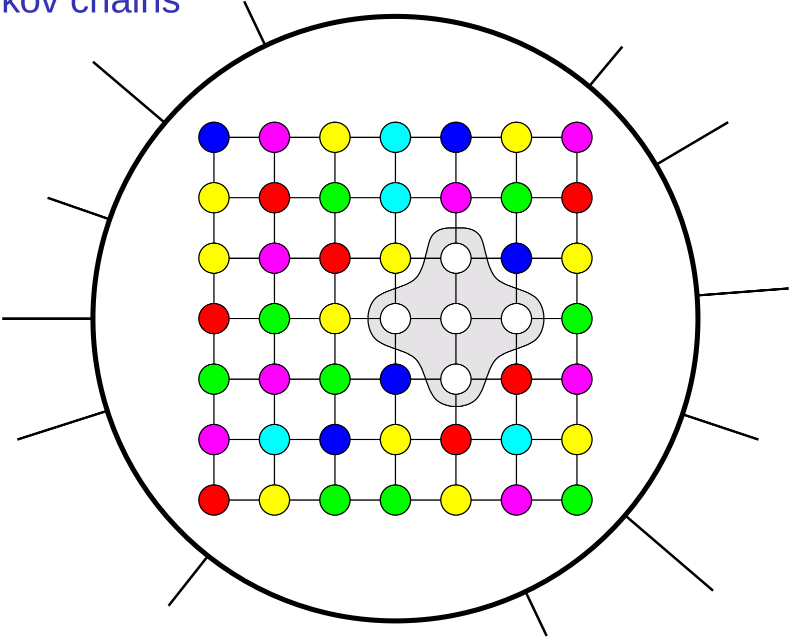
Markov chains



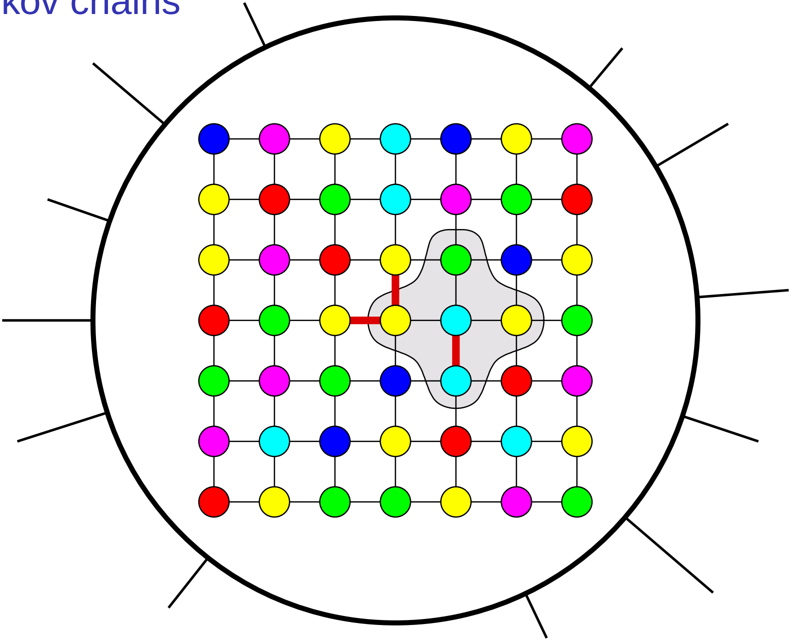
Markov chains



Markov chains



Markov chains



Markov chains

Stationary distribution

The stationary distribution of the states is identical to the distribution we want to sample colourings from.

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Total variation distance

$d_{\text{tv}}(D_1, D_2) < \epsilon$, where $\epsilon > 0$

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$d_{\text{tv}}(D_1, D_2) < \epsilon$, where $\epsilon > 0$

Polynomial number of steps (rapidly mixing)

Want number of steps be polynomial in n and $\log\left(\frac{1}{\epsilon}\right)$, where n is the size of the region.

Previous work

Theorem [L.A. Goldberg, R. Martin and M. Paterson, 2005]
For any triangle-free graph with maximum degree $\Delta \geq 3$ *we have rapid mixing if* $q \gtrapprox 1.76\Delta - 0.47$ *and* $\lambda = 0$.

Previous work

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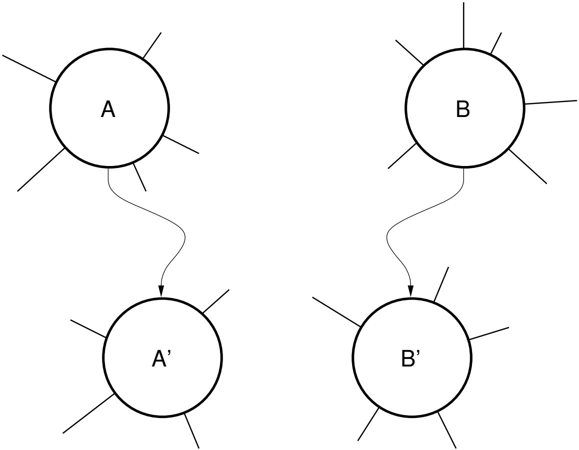
The lattice Z^2
$\Delta = 4$ so the theorem above gives rapid mixing for $q \geq 7$ and $\lambda = 0$.

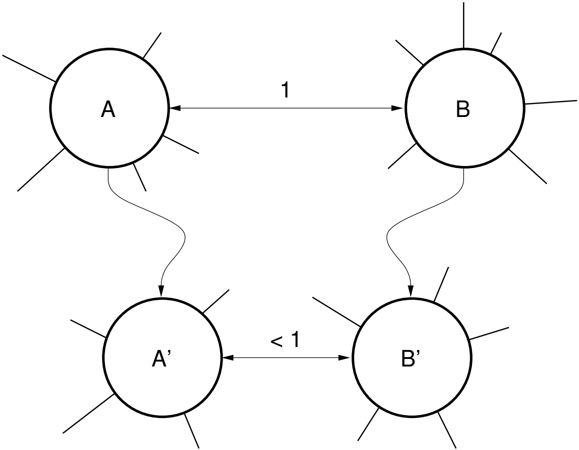
This result has now been improved [L.A. Goldberg, M. Jalsenius, R. Martin and M. Paterson, 2006].

Path coupling [R. Bubley and M. Dyer, 1997]



Path coupling [R. Bubley and M. Dyer, 1997]

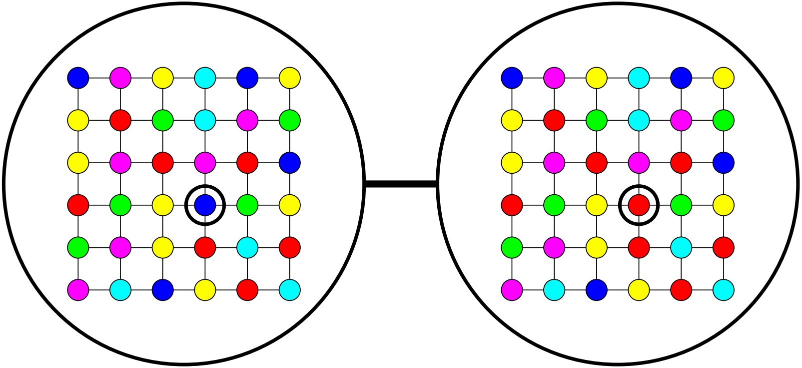
Hamming distance$(A, B) = 1$



Want the expected Hamming distance$(A', B') < 1$

Path coupling

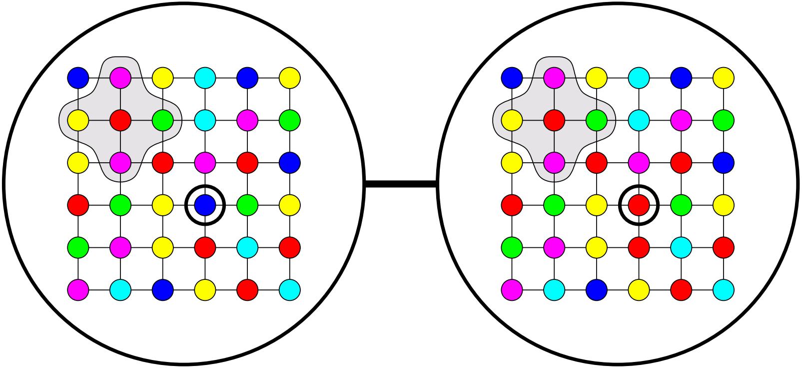
Hamming distance 1



Three scenarios can happen when applying the ball.

Path coupling

Hamming distance 1

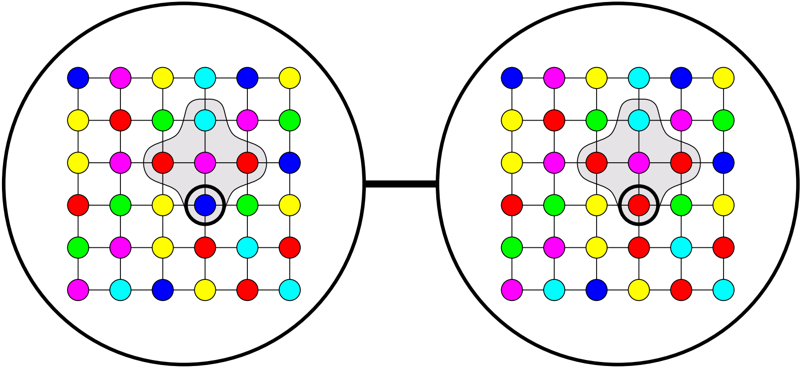


Scenario 1. The discrepancy is outside of the ball.

Hamming distance does not change.

Path coupling

Hamming distance 1

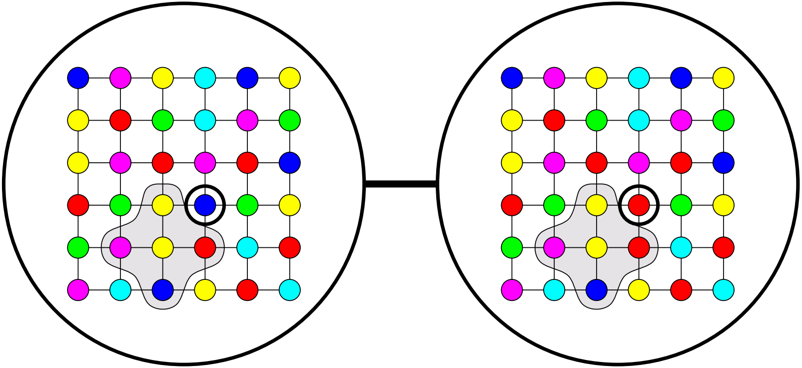


Scenario 2. The discrepancy is inside the ball.

Hamming distance drops to 0.

Path coupling

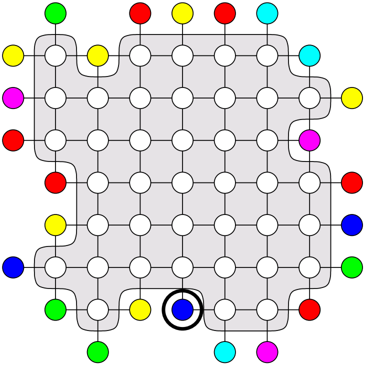
Hamming distance 1



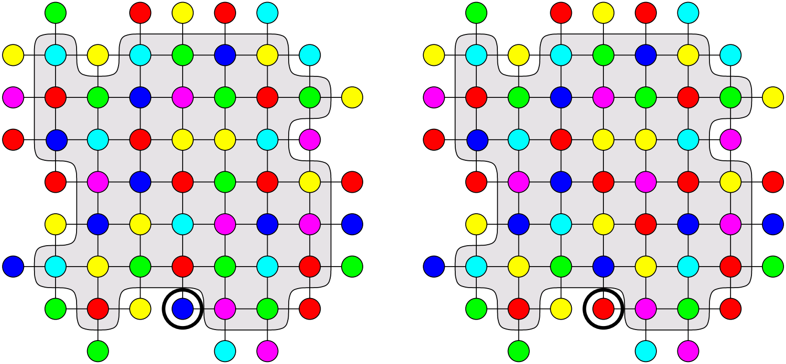
Scenario 3. The discrepancy is on the boundary of the ball.

Hamming distance can increase. How much?

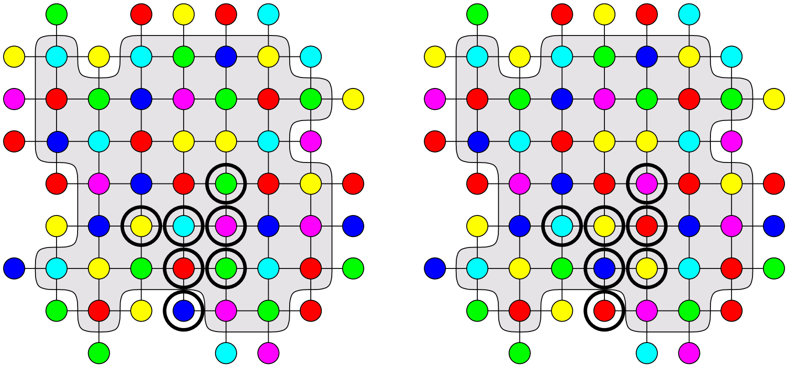
Spatial mixing



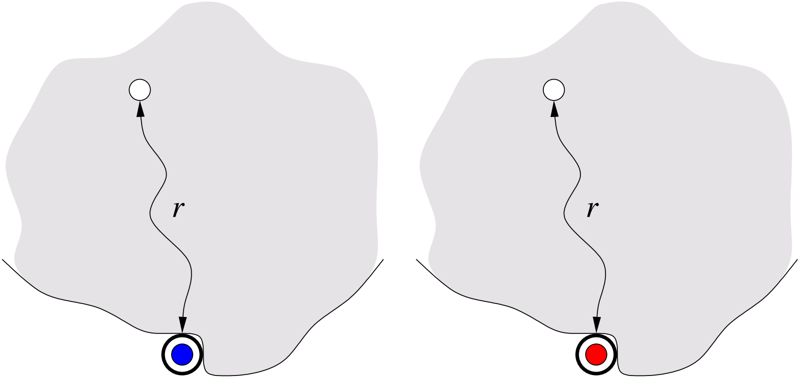
Spatial mixing



Spatial mixing



Strong spatial mixing



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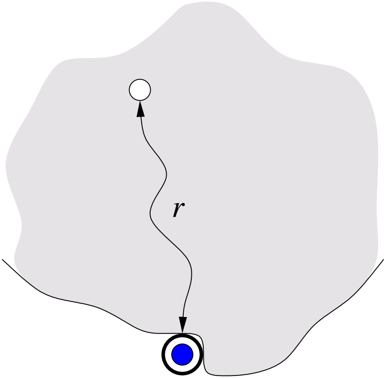

- Probability of a different colour at distance r decreases exponentially with r.

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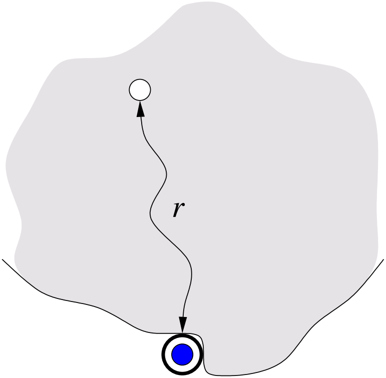
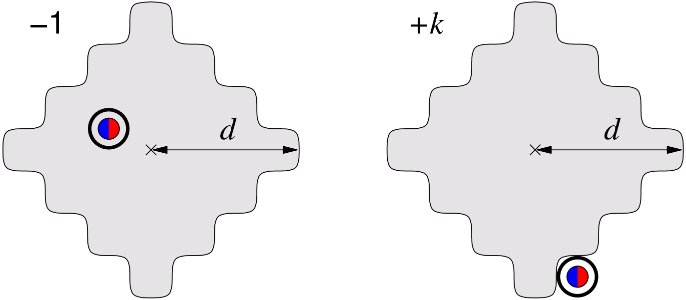


- Probability of a different colour at distance r decreases exponentially with r.
- Expected total number of introduced discrepancies in shaded region is bounded by a constant.

Path coupling



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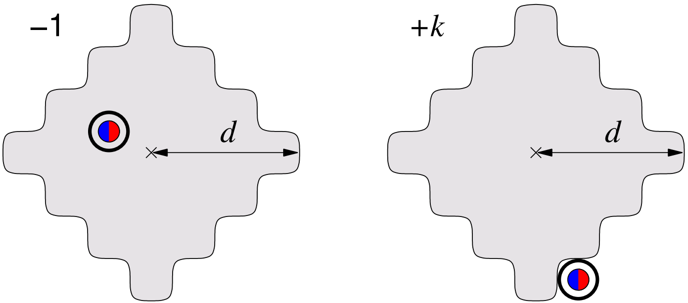
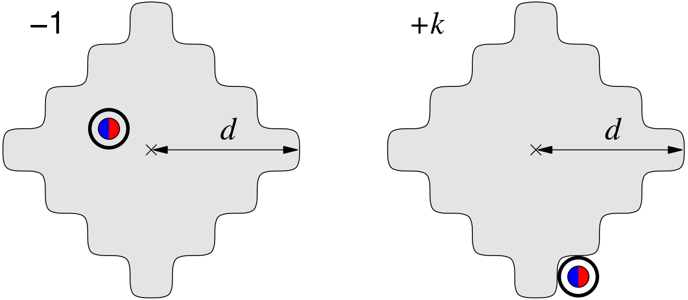


New expected Hamming distance =

$$= \quad 1 \quad - \quad 1 \times \frac{|\text{Ball volume}|}{|\text{Region}|} \quad + \quad k \times \frac{|\text{Ball boundary}|}{|\text{Region}|}$$

Path coupling



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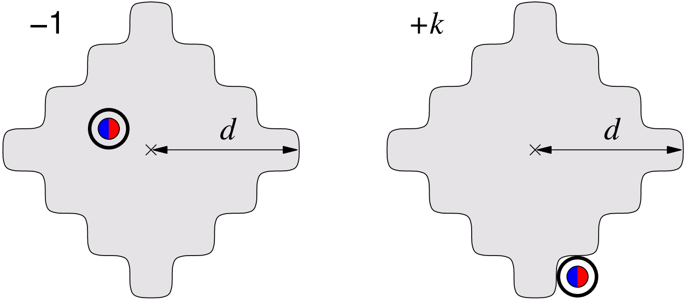

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Results

Theorem [L.A. Goldberg, M. Jalsenius, R. Martin and M. Paterson, 2006]

Consider the anti-ferromagnetic Potts model on $\mathbb{Z}^2$ with parameters q and $\lambda \leq 1$. There is strong spatial mixing in the following cases.

(i) $q \geq 6$, $\lambda \geq 0$,

(ii) $q \geq 5$, $\lambda \geq 0.127$,

(iii) $q \geq 4$, $\lambda \geq 0.262$,

(iv) $q \geq 3$, $\lambda \geq 0.393$. *(previous results: $q \geq 7$ and $\lambda = 0$)*

Corollary

The Markov chain with ball updates is rapidly mixing for the cases above, provided the radius of the ball is large enough.

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Glauber dynamics (ball updates with radius 1) is rapidly mixing for the cases above.