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Reversible Quantum Circuits from Irreversible functions

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Reversible Circuits

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- Landauer's Principle states:
  "any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment"

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  "any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment"
- Reversing a computation puts the computer back into it's initial state.

Classical Reversible Circuits

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- Wire permutations.

Classical Reversible Circuits

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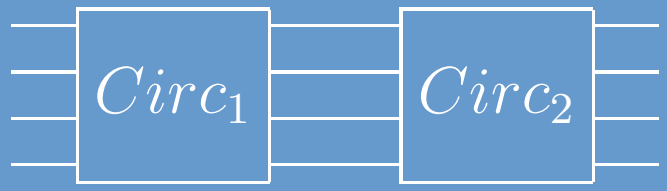
- Sequential composition.



Classical Reversible Circuits

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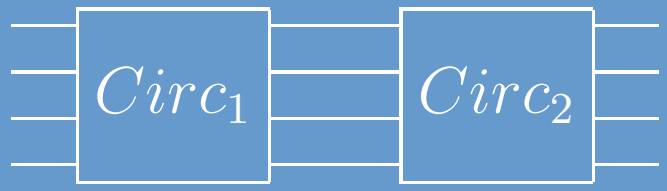
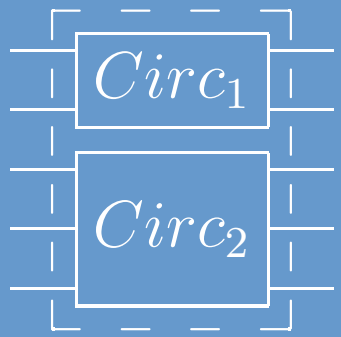


- Parallel composition.



Classical Reversible Circuits

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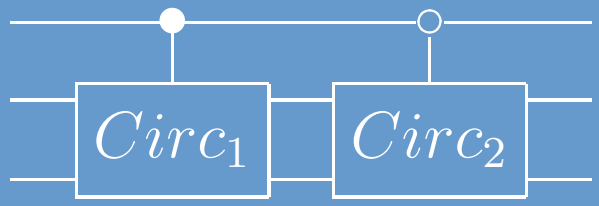
- The conditional operation.



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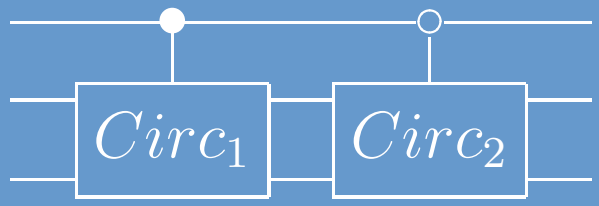
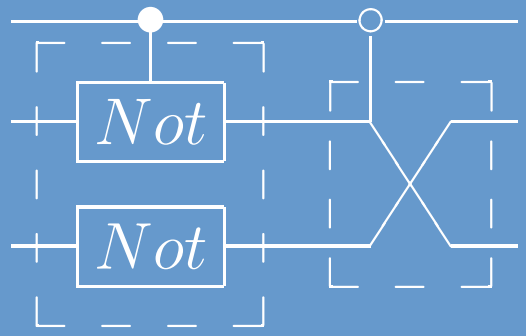
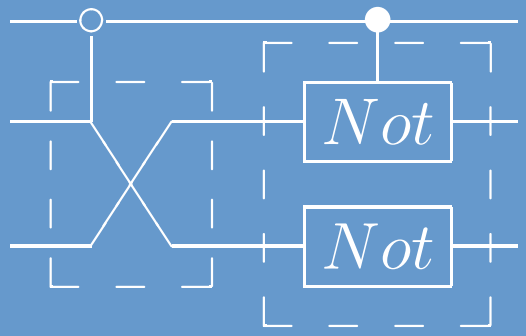


- Example:





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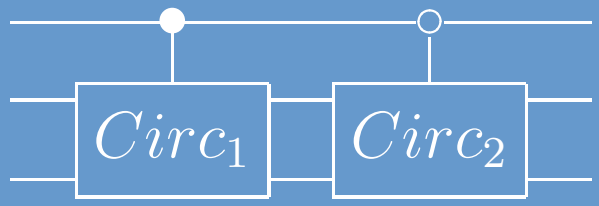

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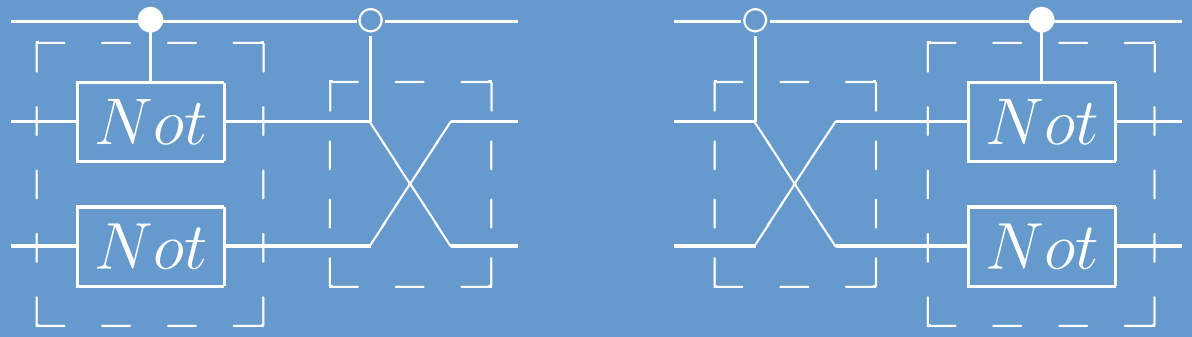

- But what about Quantum Circuits?

Classical to Quantum

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- Classical reversible circuits act on bits.

$$ReversibleCircuits \in [\,Bool\,]_n \rightarrow [\,Bool\,]_n$$

Classical to Quantum

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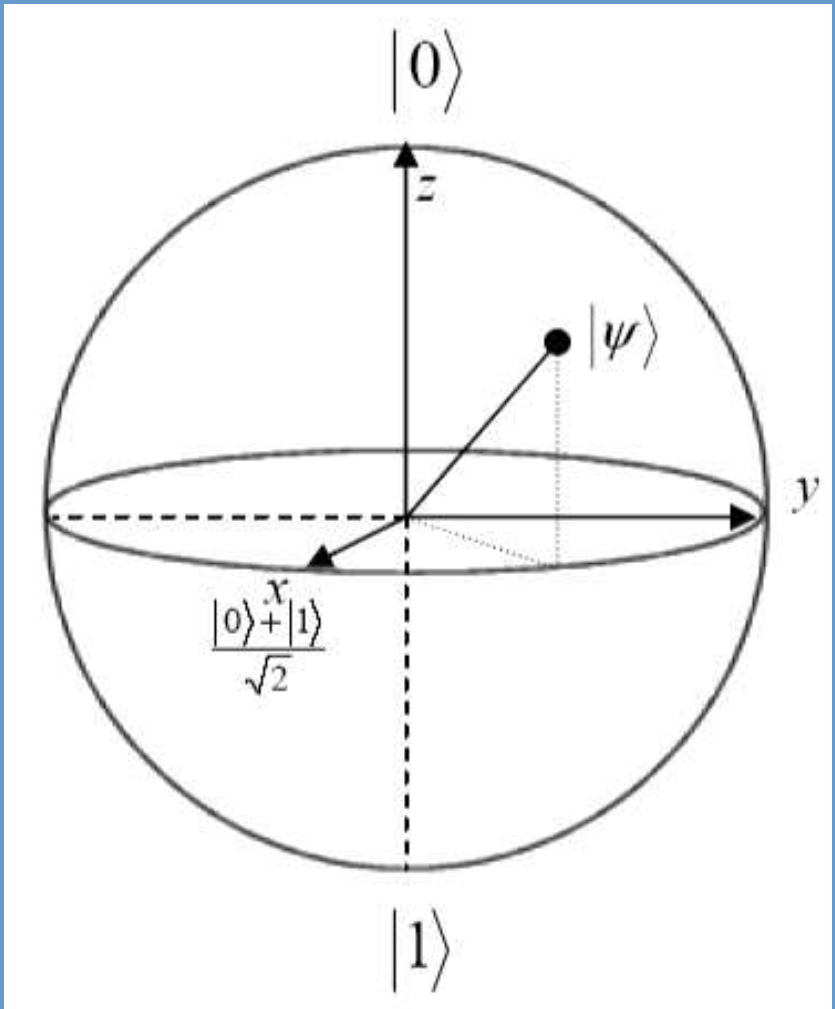
- Quantum circuits act on qubits.

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

The Bloch Sphere



Quantum Circuits

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- The classical reversible circuits are a subset of Quantum circuits.

Quantum Circuits

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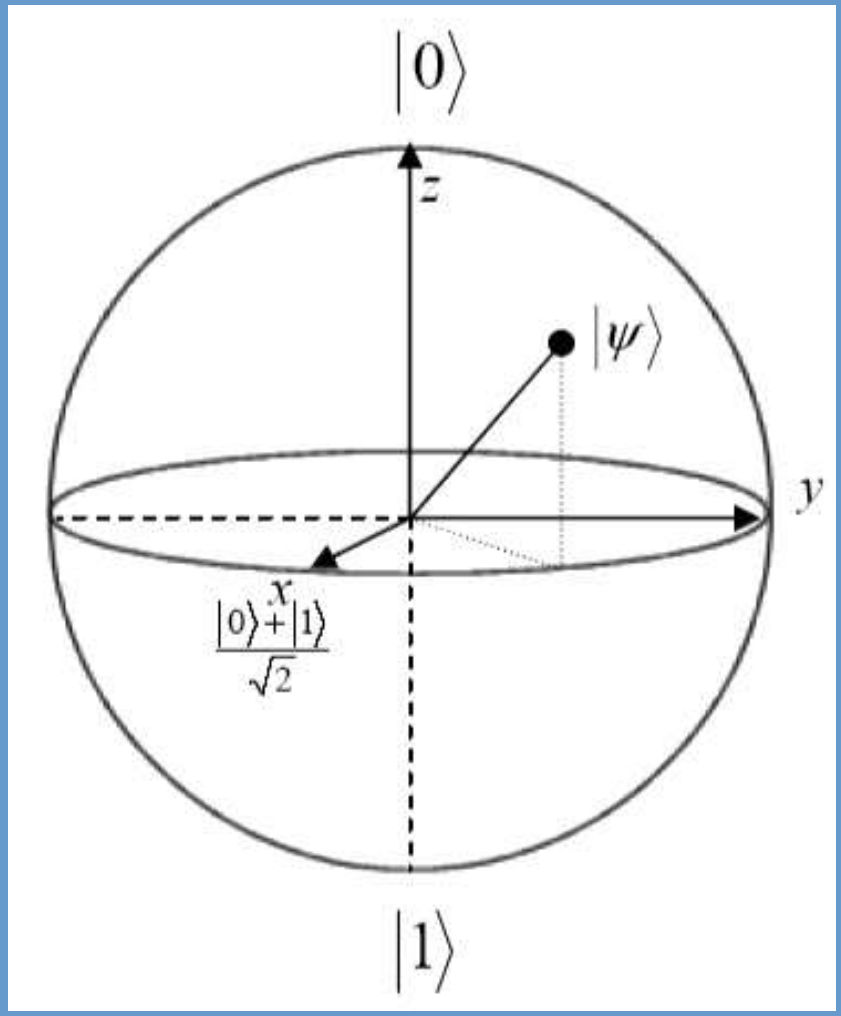
- For example the (Quantum) Not gate:

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Hadamard Transform



Irreversible Functions

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- Not all functions are logically reversible.

$$And \in (Bool, Bool) \rightarrow Bool$$
$$And\ (True, True) = True$$
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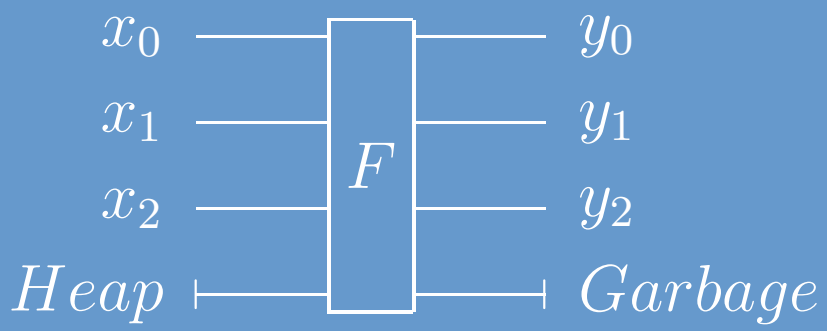
- Using Heap inputs and Garbage outputs we can create reversible circuits from irreversible functions.

Heap and Garbage

- An example Circuit:



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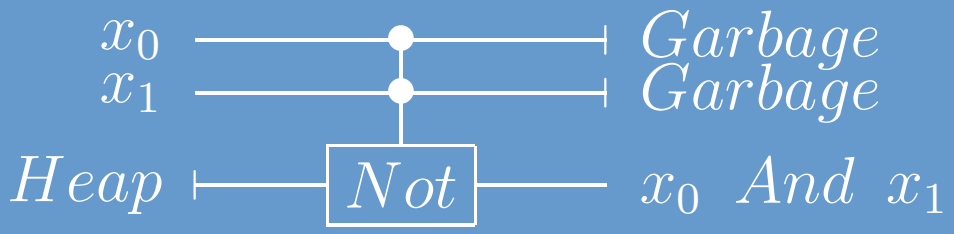
- The Garbage contains any extra information that would normally be lost by an irreversible function.
- The reversible circuit for the *And* function is: (The Toffoli gate)



Reversible "And"

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- The equivalent function for the reversible And circuit is:

$$And \in (Bool, Bool, False) \rightarrow (Bool, Bool, Bool)$$
$$And\ (True, True,\ False\ ) = (\ True\ ,\ True\ , True)$$
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- and Reversed as:

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What Now?

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- What can we do with Heap and Garbage?

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- What can we do with Heap and Garbage?
- Laws for reversible circuits with Heap and Garbage.
- That hold Classically...
- and in the Quantum case.

The 3 Laws - The First Law

- Garbage Collection

The 3 Laws - The Second Law

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- The Uselessness of Garbage Processing.

The 3 Laws - The Third Law

- The Law of Useless Heap Processing
  if $\mathrm{h}\,\vec{0} = \vec{0}$ then

Quantum or Classical

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- What are the differences for Classical and Quantum cases?

Quantum or Classical

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- What are the differences for Classical and Quantum cases?
- Measurement and Decoherence.

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- The Measurement Postulate.

The Measurement Postulate

Measurement Postulate Proof

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- The first step of the proof is to show that the following two circuits are equivalent.

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- These circuits are only made from elements in the Classical subset of the Quantum circuits

Measurement Postulate Proof

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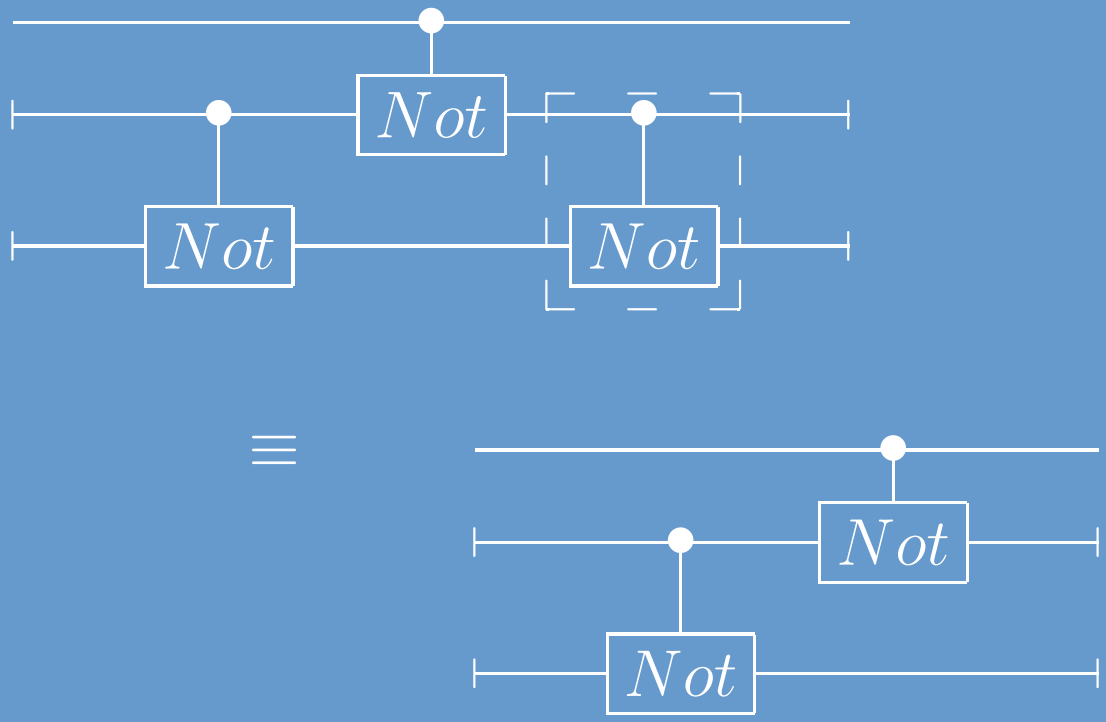
- Now we can use the Second law to eliminate the third controlled not.



Measurement Postulate Proof

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- The controlled not preserves zeros, so we can use the Third law to eliminate the first controlled not.

Measurement Postulate Proof

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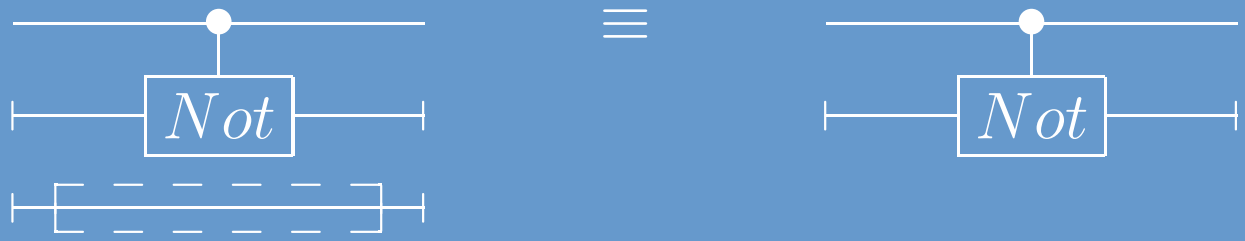
- Finally we remove the bottom wire by means of the First law.



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Summary

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- Reversible Circuits
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- Irreversible Functions
- Heap and Garbage
- The Three Laws
- The Measurement Postulate
- So what is next... ?

The End... ?

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- Future work...
  - Completeness?
  - ...

The End... ?

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- Future work...
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- Thankyou for listening.

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- Any Questions? Ask me now, find me later, or email me.
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