Recolouring Graph Colourings

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joint work with

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after an idea of

Hajo Broersma

BCTCS 2006, Swansea

Introduction

A proper vertex k-colouring of a graph $G = (V, E)$ is

- a function $\alpha : V \to \{1, 2, \ldots, k\}$
- with $\alpha(u) \neq \alpha(v)$ whenever $uv \in E$

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- nodes the proper vertex k-colourings of G
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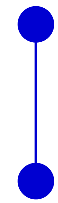
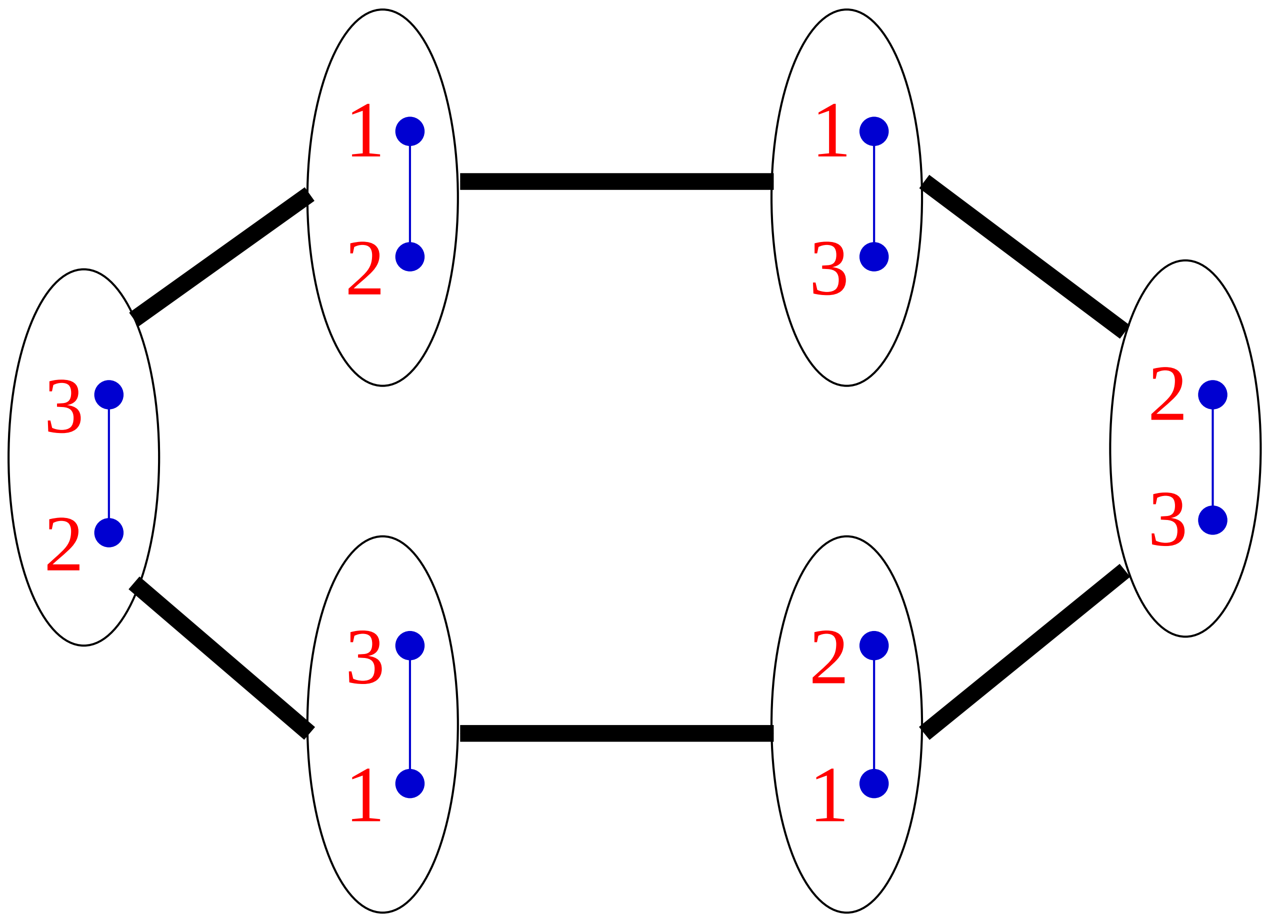
If $\mathcal{C}_k(G)$ is connected we say G is k-mixing

An example

The 3-colour graph of  is



Two decision problems

k-MIXING

Instance: k-colourable graph G
Question: Is G k-mixing?

k-COL-PATH

Instance: Graph G, k-colourings α and β of G
Question: Is there a path between α and β in $\mathcal{C}_k(G)$?

A sufficiency condition

The colouring number (or degeneracy) of a graph G is

$$\text{col}(G) = \max_{H \subseteq G} \delta(H)$$

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Proposition [Dyer, Flaxman, Frieze, Vigoda]

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Proposition [Dyer, Flaxman, Frieze, Vigoda]

- For any graph G and $k \geq \text{col}(G) + 2$, $\mathcal{C}_k(G)$ is connected

This is best possible:
there exist graphs that are not $(\text{col}(G) + 1)$-mixing

Can G be $\chi(G)$-mixing?

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Theorem

- 3-chromatic graphs are never 3-mixing
- For $\chi \geq 4$, there exist
  - χ-chromatic graphs that are χ-mixing, and
  - χ-chromatic graphs that are not χ-mixing

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- Define the weight of an edge oriented from u to v

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- For a directed cycle $\overrightarrow{C}$, define

$$W(\overrightarrow{C}, \alpha) = \sum_{\vec{e} \in \overrightarrow{C}} w(\vec{e}, \alpha)$$

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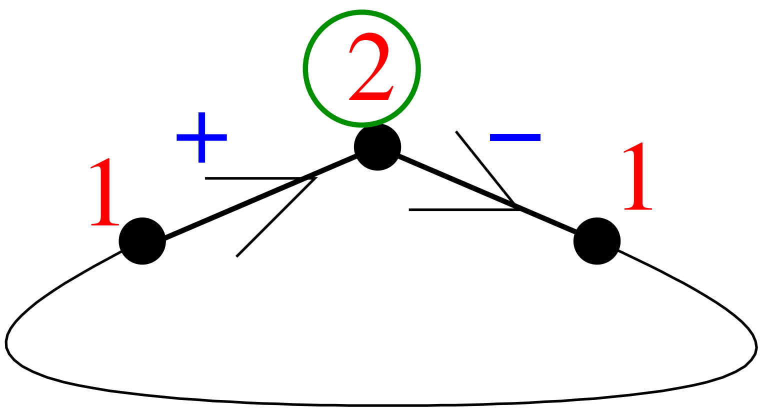
- Recolouring a vertex maintains the weight of a cycle



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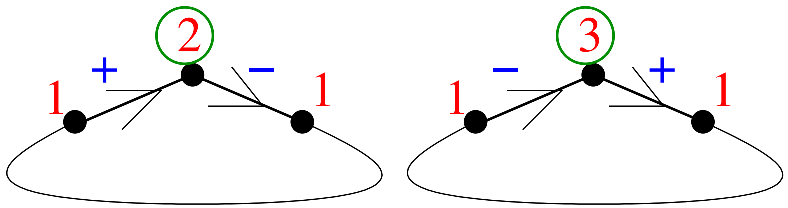

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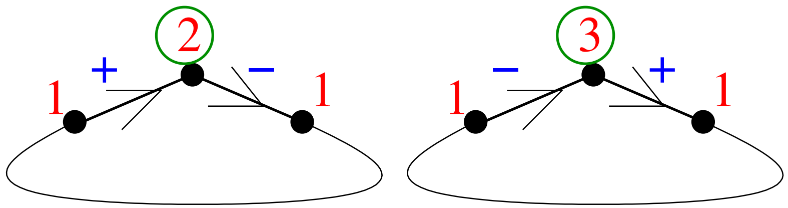

- From α, define the 3-colouring β by

$$\beta(v) = \begin{cases} 1 & \text{if } \alpha(v) = 2 \\ 2 & \text{if } \alpha(v) = 1 \\ 3 & \text{if } \alpha(v) = 3 \end{cases}$$

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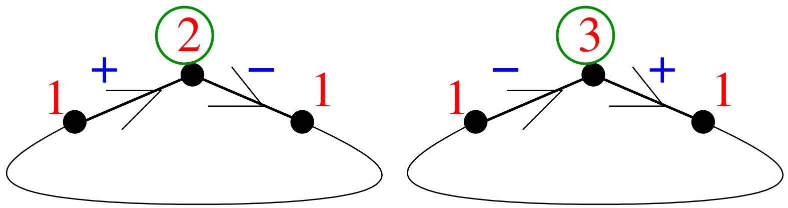

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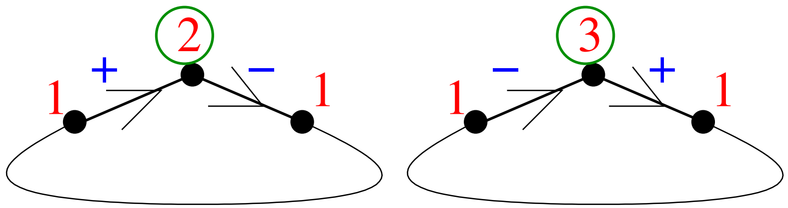

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- For C an odd cycle in G, $W(\overrightarrow{C}, \alpha) = -W(\overrightarrow{C}, \beta) \neq 0$

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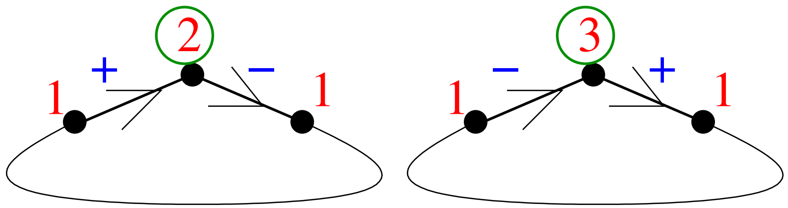

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- All edge weights change sign under β
- For C an odd cycle in G, $W(\overrightarrow{C}, \alpha) = -W(\overrightarrow{C}, \beta) \neq 0$
- Hence α and β are not connected in $\mathcal{C}_3(G)$

3-MIXING and 3-COL-PATH

- $\chi(G) = 3 \implies G$ not 3-mixing
- What about 3-MIXING for bipartite G?

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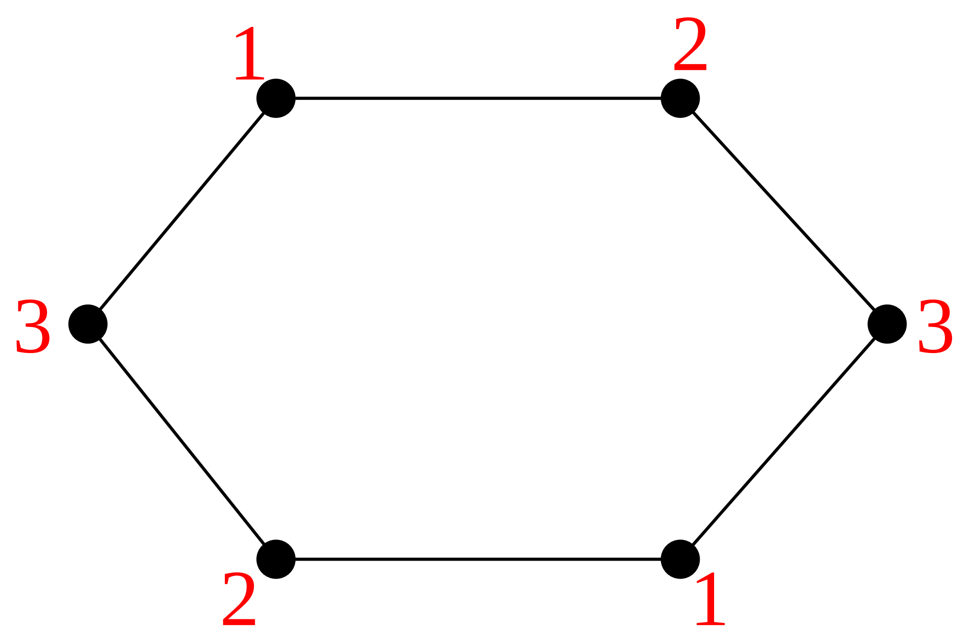

3-MIXING and 3-COL-PATH

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- For planar bipartite G, 3-MIXING $\in$ P

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Theorem

- Given two 3-colourings α, β of a graph G, it can be decided in polynomial-time whether or not α and β are connected by a path in $\mathcal{C}_3(G)$
  That is, 3-COL-PATH $\in$ P

Recolouring using extra colours

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This is best possible:
for all χ, k, with $k \geq \chi$, there exists a χ-chromatic graph with k-colourings α and β, for which $\chi - 2$ extra colours are not enough to recolour from α to β

Open Questions

- What can be said about the complexity of k-MIXING and k-COL-PATH, for $k \geq 4$?

- Is 3-MIXING coNP-complete?

- Perhaps 3-MIXING is in P?