Toughness in graphs: structural and algorithmic aspects

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Contents

Motivation

- Most of the research on toughness is based on a number of conjectures in the following paper:

  V. Chvátal,
  Tough graphs and hamiltonian circuits,
  Discrete Mathematics **5** (1973) 215–228.

- We will discuss most of the conjectures and the progress made towards proving or refuting them.
- More details can be found in a survey paper by Bauer, B. and Schmeichel that will appear in the next issue of Graphs and Combinatorics.
- The survey contains more than 160 references to papers related to toughness.

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- Hence if G is not complete, $\tau(G) = \min\{|S|/\omega(G-S)\}$, where the minimum is taken over all cut sets of vertices in G.

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- It is not difficult to check that a hamiltonian graph is 1-tough.
- Is there a level of toughness that guarantees the existence of k-factors or Hamilton cycles?

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- He observed that the validity for $k = 1$ follows from Tutte's matching theorem.
- All questions were answered by the following two results.

Chvátal's conjecture is true and sharp

Theorem (Enomoto, Jackson, Katerinis, Saito '85)

Let G be a k-tough graph on n vertices with $n \geq k+1$ and kn even. Then G has a k-factor.

Theorem (Enomoto, Jackson, Katerinis, Saito '85)

Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k-\epsilon)$-tough graph G on n vertices with $n \geq k+1$ and kn even which has no k-factor.

Toughness and hamiltonicity

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- As we have seen, Enomoto et al. have found $(2-\epsilon)$-tough graphs having no 2-factor for arbitrary $\epsilon > 0$.
- So what about 2-tough graphs?
- Another conjecture of Chvátal was that every 2-tough graph in which every vertex has a connected neighbourhood, is hamiltonian.

The 2-tough conjecture

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- The truth of this 2-tough conjecture would imply the well-known result of Fleischner that the square of any 2-connected graph is hamiltonian.
- Moreover, it would imply the truth of two other conjectures that have been open for about twenty years:
  - Every 4-connected line graph is hamiltonian (Thomassen).
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- However, it turns out that not all 2-tough graphs are hamiltonian.

Counterexamples

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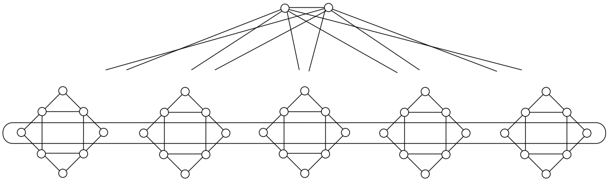

Figure: A 2-tough nonhamiltonian graph

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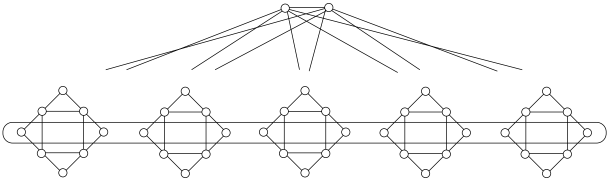

Figure: A 2-tough nonhamiltonian graph

All counterexamples are neighbourhood-connected, so the weaker conjecture of Chvátal is also false.

The t_0-tough conjecture

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  - There exist $(\frac{7}{4} - \epsilon)$-tough chordal nonhamiltonian graphs (Bauer, B., Veldman).
  - Every t-tough planar chordal graph with $t > 1$ is hamiltonian (Böhme, Harant, Tkáč).

Degree conditions

- Many known sufficient conditions for the existence of a Hamilton cycle are based on the degrees of the vertices of a graph.
- The earliest ones are due to Dirac and to Ore:

Theorem (Dirac '52)

Let G be a graph on $n \geq 3$ vertices with $\delta \geq \frac{n}{2}$. Then G is hamiltonian.

Theorem (Ore '60)

Let G be a graph on $n \geq 3$ vertices with $\sigma_2 \geq n$. Then G is hamiltonian.

- These conditions can be weakened if one imposes a toughness condition. We give a few examples.

Weakened degree conditions

Theorem (Jung '78)

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Theorem (Bauer, B., Van den Heuvel, Veldman '95)

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- This implies that Chvátal's t_0-tough conjecture is true within the class of graphs having $\delta(G) \geq \epsilon n$, for any fixed $\epsilon > 0$.

Toughness and pancyclicity

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Theorem (Bauer, Van den Heuvel, Schmeichel '95)

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- Subsequently, Alon proved a stronger result.

Theorem (Alon '95)

For every t and g there exists a t-tough graph of girth greater than g.

Toughness and complexity

- The problem of determining the complexity of recognizing t-tough graphs was first raised by Chvátal, but not in the original paper.
- The question later appeared in a paper by Thomassen '81.
- Consider the following decision problem, where t is any positive rational number.

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QUESTION : Is $\tau(G) \geq t$?

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- The complexity was settled about ten years later:

Theorem (Bauer, Hakimi, Schmeichel '90)

For any positive rational number t, t-TOUGH is NP-hard.

Special graph classes

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- For example, Matthews and Sumner '84 have shown that for claw-free graphs, $\tau = \kappa/2$.
- Hence the toughness of claw-free graphs, and consequently of line graphs, can be determined in polynomial time.
- Thus, while it is NP-complete to determine whether a line graph is hamiltonian (Bertossi '81), it is polynomial to determine whether a line graph is 1-tough.

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Theorem (Woeginger '98)

For any rational number $t \geq 0$, the class of t-tough split graphs can be recognized in polynomial time.

Other graph classes

- For many subclasses of graphs, however, it is NP-hard to recognize t-tough graphs, e.g., for graphs having minimum degree "almost" high enough to ensure that they are t-tough:

Theorem (Bauer, Morgana, Schmeichel '94)

Let $t \geq 1$ be a rational number. If $\delta \geq (\frac{t}{t+1})n$, then G is t-tough. On the other hand, for any fixed $\epsilon > 0$, it is NP-hard to determine whether G is t-tough for graphs G with $\delta \geq (\frac{t}{t+1} - \epsilon)n$.

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Theorem (Kratsch, Lehel, Müller '96)

1-TOUGH remains NP-hard for bipartite graphs.

- Consequently, 1-TOUGH is also NP-hard for the larger class of triangle-free graphs.

Regular graphs

- An important class of graphs that has received considerable attention is the class of regular graphs. Note that the maximum possible toughness of an r-regular noncomplete graph is $r/2$.
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- Chvátal asked for which values of r and $n > r + 1$ there exists an r-regular, $r/2$-tough graph on n vertices.
- He observed that this is always the case for r even.
- He also conjectured that for r odd and n sufficiently large, it would be necessary that $n \equiv 0 \mod r$, and verified this for $r = 3$.

Regular graphs

- An important class of graphs that has received considerable attention is the class of regular graphs. Note that the maximum possible toughness of an r-regular noncomplete graph is $r/2$.
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- But for all odd $r \geq 5$, Doty '91 and Jackson and Katerinis '94 independently constructed an infinite family of r-regular, $r/2$-tough graphs on n vertices with $n \not\equiv 0 \mod r$.

Complexity for regular graphs

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- The complexity of recognizing r-regular, t-tough graphs remains completely open when $2t < r < 3t$, and the complexity when $r = 2t + 1$ seems especially intriguing.

Conclusions and open problems

- Since Chvátal introduced toughness in 1973, much research has been done that relates toughness conditions to the existence of cycle structures.
- In our survey paper, we have gathered many of the important results in this area.
- Historically, the motivation for this research was based on a number of conjectures in Chvátal's paper.
- The most challenging of these conjectures is still open:

  Is there a finite constant t_0 such that every t_0-tough graph is hamiltonian? If so, what is the smallest such t_0?

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- However Matthews and Sumner '84 have conjectured that 4-connected (2-tough) claw-free graphs are hamiltonian.

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- The gaps in our knowledge for claw-free and chordal graphs imply a number of challenging open problems.
- The same is true for the class of triangle-free graphs.

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- It even appears that a class of triangle-free graphs with no 2-factor constructed by Bauer et al. '96 has toughness approaching 2 from below.
- These examples suggest the intriguing possibility that every 2-tough triangle-free graph is hamiltonian.
- On the other hand, it remains completely open whether the t_0-tough conjecture holds for the class of triangle-free graphs.

Degree conditions

- We know that the t_0-tough conjecture is true within the class of graphs on n vertices satisfying $\delta \geq \epsilon n$, for any fixed $\epsilon > 0$.

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- We know that if G is a 2-tough graphs on n vertices with $\delta \geq n/3$, then G is hamiltonian.
- What if $5 \leq \delta < n/3$?
- The early research on toughness and cycle structure concentrated on sufficient degree conditions which, combined with a certain level of toughness, would yield the existence of long cycles.
- The survey paper contains a wealth of results in this direction.
- One of the major open problems in this area is the conjecture that every 1-tough graph on n vertices with $\sigma_3 \geq n \geq 3$ has a cycle of length at least $\min\{n, (3n+1)/4 + \sigma_3/6\}$.

Other open problems

- Another interesting problem is to find the best possible minimum degree condition to ensure that a 1-tough triangle-free graph is hamiltonian.
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- Another interesting problem is to find the best possible minimum degree condition to ensure that a 1-tough triangle-free graph is hamiltonian.
- We know the answer lies somewhere between $(n+2)/4$ and $(n+1)/3$.
- If we do not impose a degree condition, toughness conditions can still guarantee cycles of length proportional to a function of the number of vertices of the graph.
- Two of the most challenging open problems in this area are whether there exist positive constants A and B, depending only on t, such that every 2-connected, respectively 3-connected, t-tough graph on n vertices has a cycle of length at least $A \log n$, respectively n^B.
- Both problems have affirmative solutions for planar graphs.

Toughness and factors

- Another area of research has involved finding toughness conditions for the existence of certain factors in graphs.
- One of the challenging open problems in this area is to determine whether every 3/2-tough maximal planar graph has a 2-factor.
- If so, are they all hamiltonian?
- We also do not know whether every 3/2-tough planar graph has a 2-factor.

Computational complexity

- Research on toughness has also focused on computational complexity issues.
- In particular, we now know that recognizing t-tough graphs is NP-hard in general, whereas it is polynomial within the class of claw-free graphs and within the class of split graphs.
- For many other interesting classes, this complexity question is still open, e.g., for (maximal) planar graphs and for chordal graphs.

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- For many other interesting classes, this complexity question is still open, e.g., for (maximal) planar graphs and for chordal graphs.
- Within the class of r-regular graphs with $r \geq 3t$, recognizing t-tough graphs has been shown to be NP-hard.
- The problem is trivial if $r < 2t$, but its complexity is open for values of r with $2t \leq r < 3t$.
- It was conjectured by Goddard and Swart '91 to be polynomial for $r = 2t$, and seems especially interesting when $r = 2t + 1$.

Main conclusion

There are still many tough problems to be solved!!