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by

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Modelling Filament Stretching Flows with Strain-hardening Models and Sub-cell Approximations[†]

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Abstract

This article analyses the transient viscoelastic response of strain-hardening fluids in filament stretching flows. We utilise an Arbitrary Lagrangian/Eulerian temporal approach (ALE), coupled with a particle-tracking procedure for free-surface movement and a hybrid finite volume/element method upon the domain. We are able to contrast findings between Oldroyd, Giesekus and linear Phan-Thien/Tanner models, and distinguish between single and multi-mode implementations. In this manner, we identify the impact that greater severe strain-hardening has in this transient flow context. Contributions from shear-thinning rheology may be gathered in particular by comparing single-mode solution response between a shear-thinning Giesekus and a constant shear viscosity Oldroyd-B model. A parameter study on inertial and surface tension effects has been undertaken, where we isolate the occurrence of asymmetries in the flow under certain conditions, leading to the onset and formulation of bead-like structures. This elucidates the specific localised influence that surface tension and gravitational forces have upon some stretching filament flows.

Keywords: Strain-hardening, filament stretching, single and multi-mode, ALE, surface tension, inertia

1. Introduction

Filament stretching may be undertaken in a variety of different ways. In the procedure of current interest, a cylindrical liquid column is first generated between two concentric circular plates, and then, elongated by pulling one or both of the end-plate fixtures apart at an exponentially increasing rate. In principle, the resulting flow kinematics in the liquid column should approximate ideal uniaxial elongational flow. Under such circumstances, the desired extensional viscosity function may be determined from the axial force at the end-plates. This force is measured as a function of time, and the Hencky strain applied to the material is computed from the total stretch imposed on the sample.

The principal practical complications that arise in filament stretching are related to two aspects, free-surface deformation and non-deforming end-plates. During free-surface deformation necking occurs due to the balance of forces that constitute the pertaining surface tension. The presence of non-deforming end-plates causes significant shear at the feet of the filament due to the imposed no-slip boundary conditions. This leads to filament deformation, dominated by extension, but still with some presence of shear flow. To tackle these issues, we invoke particular techniques specifically designed to govern free-surface deformation under these circumstances.

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Strain-hardening like response is an issue of particular relevance in this type of work. Some typical aspects through which one may observe the influence of strain-hardening are: the rapid increase in tensile stress encountered; the adjustment in slope governing the time change of the filament mid-point radius; and the tendency towards axial uniformity in the distribution of the filament radius. We comment that all of these features are detected and the form of strain-hardening behaviour predicted in the present investigation is found to agree well with that in the open literature.

There are many nonlinear constitutive models available today that may describe the rheology of interest here. We choose a Giesekus model as our base-case [1], in both single and multi-mode form, and contrast this against Oldroyd and Phan-Thien-Tanner (PTT) models. The Giesekus model can fit both linear and nonlinear shear rheology of most concentrated polymeric solutions, and is commonly employed to represent weakly strain-hardening fluids. For example in this respect, the Giesekus model has already been employed by various authors, such as Li et al. [2] and Yao et al. [3]. In their work, Li et al. utilised a higher-order-discrete-elastic-viscous-stress (*hp-DEVSS*) finite element method to solve an axisymmetric stagnation flow. In contrast and using the commercial software package POLYFLOW, Yao et al. [3] investigated the transient viscoelastic behaviour in filament stretching devices under uniaxial elongation, following the cessation of stretching.

The numerical solution of viscoelastic filament stretching flows has been studied in a number of instances covering a variety of different approaches, calling upon Lagrangian and Eulerian treatments with integral and differential constitutive laws [4-7]. For example, Sizaire and Legat [4] considered the two-dimensional finite element simulation of filament stretching by employing a differential FENE-CR model to substantiate strain-hardening/constant shear viscosity material response in an Eulerian configuration (L^2 extensibility coefficient of 4325.5). Free-surface movement was determined through the use of a surface height function 'h', defined as the normal displacement to the initial surface position. This work employed a moving grid algorithm, in conjunction with a Thompson conformal mapping, to avoid distorted elements. With this approach, FENE-CR field results were presented up to Hencky-strains of 2.56, reportedly failing earlier than equivalent experiments. An alternative formulation is that adopted in the work of Hassager and co-authors [5-7], where a Lagrangian treatment is preferred, coupled to an integral constitutive relationship. Typically, in the article of Bach et al. [5] the non-linear constitutive equation chosen to represent the fluid was a slight modification of the K-BKZ version of the molecular stress function (MSF) model of Wagner et al. [8], which describes a range of strain-hardening response. This Lagrangian approach demands remeshing and solution reprojection periodically throughout the filament evolution process. Such work is pertinent to the present study through its attention to the growth of non-axisymmetric/three-dimensional disturbances in the flow (physical instability), and circumstances of both failure and non-failure within polymeric liquid filaments.

In the present study, we employ a novel hybrid finite volume/element scheme ($h_v\text{-}fV$) developed in [9] and applied here specifically within the transient viscoelastic free-surface context. The scheme is centred about a number of key features. First, it is a time-stepping procedure of fractional-stage form on each Hencky-strain step, uniting incremental pressure-correction stages with Lax-Wendroff/Taylor-Galerkin time-splitting. Second, an Arbitrary-Lagrangian-Eulerian (*ALE*) formulation is preferred to a pure Eulerian alternative, where we appeal to a *Compressed-Mesh (CM)* spatial implementation as opposed to that of a *Volume-of-Fluid (VOF)* scheme. Third, our previous studies have identified the superiority of particle-tracking (dx/dt) over height function ($\partial h/\partial t$) schema to determine the motion of the filament free-surface. Fourth, domain spatial discretisation is based on a finite volume sub-cell approximation for stress, with finite element technology invoked for velocity and pressure.

Regarding *ALE* implementations, these are frequently employed schemes to track complex free-surface shapes that retain freedom of mesh movement. In such procedures, the mesh is shifted with a suitable mesh velocity. The *ALE*-formulation was introduced in the finite difference domain by Noh [10] and Hirt et al.[11]. This was further extended into the finite element domain by Hughes et al. [12] for incompressible viscous flows.

Following Yao et al. [13], we proceed to analyse the rheological response of a range of different models in this complex extensional flow setting - of Oldroyd-B, Giesekus and Phan-Thien/Tanner models. The *linear* PTT version (LPTT) is useful on account of its larger transient extensional response than that associated with the Giesekus model, hence exposing the impact of this particular dynamical feature. We focus primarily on single-mode approximations, with a Weissenberg number (We) chosen as 1.89, and an initial filament aspect-ratio (Λ_0) of 0.54. We demonstrate how we have been able to reach higher Hencky-strain levels above *three* and pertaining to *four plus* units. Furthermore, we extend the numerical scheme to accommodate a multi-mode implementation, relying upon our previous PTT multi-mode experience derived for steady-state wire-coating flows with three to seven-modes [14, 15].

Within the larger Hencky-strain range, $\epsilon > 3.0$, we are able to investigate the specific, competing and segregated impact that inertial-gravitational forces and surface tension forces have upon evolving filament forms and deformation-states. In this respect, we are able to isolate the onset of asymmetries and depict the growth of single to multiple bead-like structures, tying this down to the rheological properties of the fluids in question. Others have observed such phenomena [16-22]. Bazilevsky et al. [17], Stelter et al. [18], and Tripathi et al. [19] reported studies using the capillary-induced thinning of a liquid filament as a rheometric device for quantifying the behaviour of complex extensional flows. The liquid bridge formed between two coaxial cylindrical plates was extended beyond its static stability limit by subjecting it to a uniaxial step-strain. Subsequently, the liquid-bridge underwent a capillary thinning process, which eventually lead to break-up, due to the combined effects of capillary pressure, gravity, viscous and elastic stresses. The temporal evolution of filament radius was noted employing high-speed video imaging. Entov and Hinch [20] conducted a detailed study of the evolution of a viscoelastic fluid undergoing capillary-driven break-up. Findings from this and the above mentioned studies point to the fact that during the course of stretching, there can be a period when viscous and gravitational forces are negligible, whilst elastic and capillary forces balance one another. During this interval, the local extension-rate at the neck of the liquid-bridge is constant and the filament radius decreases exponentially in time. Under such circumstances, an estimate for the characteristic relaxation time of the fluid may be determined based upon the measurement of the rate of decrease of the filament radius. In this respect, the data obtained by Anna and McKinley [16] using extensional rheometers has corroborated quantitatively with the observations of Entov and Hinch [20]. On this body of evidence, we proceed to conduct the present investigation.

2. Governing Equations

The governing equations for incompressible, viscoelastic flows are represented by conservation laws for mass and momentum, in conjunction with an equation of state for stress. Posed in dimensional form, the balance in field equations under isothermal flow conditions may be expressed as,

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\rho \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \nabla \cdot (2\mu_s \mathbf{d} + \boldsymbol{\tau}) + \rho \mathbf{g}. \quad (2)$$

Here, ρ , $\mathbf{u}$, p , $\boldsymbol{\tau}$ represent the fluid density, the velocity, the hydrodynamic pressure and the extra-stress tensor, respectively. The gravity is represented by $\mathbf{g}$. The total viscosity μ_0 is segregated into a solvent (μ_s) and a polymeric (μ_p) contribution, so that $\mu_0 = \mu_s + \mu_p$; $\mathbf{d} = (\mathbf{L} + \mathbf{L}^T)/2$ represents the rate-of-deformation tensor and $\mathbf{L}^T = \nabla \mathbf{u}$, the velocity gradient.

The relevant constitutive equations for Giesekus [1] and LPTT [23, 24] models may be represented through a generic form as:

$$\lambda \frac{\partial \boldsymbol{\tau}}{\partial t} = -\lambda \mathbf{u} \cdot \nabla \boldsymbol{\tau} - f \boldsymbol{\tau} + 2 \frac{\mu_1}{\mu_0} \mathbf{d} + \lambda (\mathbf{L} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}^T) + \lambda \xi \{ \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{D} \} - \frac{\alpha \lambda}{\mu_1 / \mu_0} \boldsymbol{\tau} \cdot \boldsymbol{\tau}, \quad (3)$$

where, the fluid is defined through its polymeric viscosity component (μ_p) and relaxation time (λ). The Giesekus model nonlinear mobility factor parameter- (α) is associated with anisotropic material response and to be physically realistic, must satisfy $0 \leq \alpha \leq 0.5$. The remaining parameters ($f=1$ and $\xi=0$) are related to the LPTT model. In contrast, the mobility factor should vanish to recover the LPTT model, and f and ξ are defined viz,

$$f = 1 + \frac{e\lambda}{\mu_1} \text{tr}(\boldsymbol{\tau}), \quad e \geq 0 \quad \text{and} \quad 0 \leq \xi \leq 2.$$

To non-dimensionalise the governing set of equations for continuity, momentum and constitutive law, we employ scales over velocity (U) and length (L), which transform variable reference on velocity, pressure, length, stress, viscosity and time, as follows:

$$u = u^* U, \quad p = \mu_0 \frac{U}{L} p^*, \quad x = x^* L, \quad \boldsymbol{\tau} = \mu_0 \frac{U}{L} \boldsymbol{\tau}^*, \quad \mu = \mu_0 \mu^*, \quad t = \frac{L}{U} t^*. \quad (4)$$

Discarding asterisk notation for clarity and introducing non-dimensional group numbers provides the associated non-dimensional system,

$$\nabla \cdot \mathbf{u} = 0, \quad (5)$$

$$Re \frac{\partial \mathbf{u}}{\partial t} = -Re \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \nabla \cdot (2 \frac{\mu_s}{\mu_0} \mathbf{d} + \boldsymbol{\tau}) + \frac{Bo}{Ca}. \quad (6)$$

$$We \frac{\partial \boldsymbol{\tau}}{\partial t} = -We \mathbf{u} \cdot \nabla \boldsymbol{\tau} - f \boldsymbol{\tau} + 2 \frac{\mu_1}{\mu_0} \mathbf{d} + We (\mathbf{L} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}^T) + We \xi \{ \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{D} \} - \frac{\alpha We}{\mu_1 / \mu_0} \boldsymbol{\tau} \cdot \boldsymbol{\tau}. \quad (7)$$

with $f = 1 + \frac{eWe}{\mu_1} \text{tr}(\boldsymbol{\tau})$ and group numbers of Reynolds, Weissenberg, Bond and Capillary number, defined viz.,

$$Re = \frac{\rho UL}{\mu_0}, \quad We = \frac{\lambda U}{L}, \quad Bo = \frac{\rho g L^2}{\chi}, \quad Ca = \frac{\mu_0 U}{\chi}, \quad (8)$$

with a surface tension coefficient, χ . For initial conditions, we adopt a quiescent state on field variables and take ambient surrounding pressure level as p_a ,

$$\mathbf{u}(r, z) = \mathbf{0}, \quad \boldsymbol{\tau}(r, z) = \mathbf{0} \quad \text{and} \quad p(r, z) - p_a = 0 \quad \text{at} \quad t \leq 0^-,$$

with the imposed initial stretch-rate ($\dot{\epsilon}_0$) and filament length (L_0). The initial aspect ratio of the liquid-bridge is defined as $\Lambda_0 = L_0 / R_0$, R_0 being the initial filament radius. Following reference [3] and to furnish equitable comparison, we have chosen in this work to limit attention throughout to a single filament aspect-ratio, $\Lambda_0 = 0.54$ (see [9] for alternative choices).

With respect to boundary conditions, we consider: no-slip along the interface between the liquid and end-plates, and impose axisymmetry along the filament-central axis (axial). End-plate movement is prescribed in exponential form, to establish a constant stretch-rate at the filament centre, viz,

$$U_z(t) = \pm(\dot{\epsilon}_0 L_0 / 2) \exp(\dot{\epsilon}_0 t), \quad U_r(t) = 0.$$

For the multi-mode instance with n -modes, the constitutive equation (7) must be solved for each individual mode (i), with its partial viscosity (μ_i), relaxation time (λ_i) and associated We_i (definition as above, but on λ_i). Then, the total stress contributing to the momentum equation,

$$\boldsymbol{\tau} = \sum_{i=1}^n \boldsymbol{\tau}_i, \text{ is composed from the sum of the individual stress modes } (i=1, n).$$

3. Numerical procedures

3.1 Free-surface computation/ALE implementation

A particle-tracking technique is utilised to compute free-surface movement. The fluid surface is assumed to be composed of a series of distributed particles, located at nodes from the start of a time-step. Typically, a fluid particle is shifted according to an Euler procedure:

$$\mathbf{X}_i^{n+1} = \mathbf{X}_i^n + \Delta t \mathbf{u}(\mathbf{X}_i^n, t^n)$$

where, for a particle at location i on the free-surface in positions ($\mathbf{X}_i^n, \mathbf{X}_i^{n+1}$) at times

(t^n, t^{n+1}), its particle-velocity is $\mathbf{u}(\mathbf{X}_i^n, t^n)$ at $\mathbf{X}_i^n$ and time t^n . In this fashion, particles and hence, mesh-nodes are shifted to their updated end-of-time-step position,

$$r^{n+1} = r^n + u_r \Delta t \quad \text{and} \quad z^{n+1} = z^n + u_z \Delta t.$$

By employing this particular strategy, mesh-nodes are shifted both in radial and axial directions. This lies in stark contrast to our earlier approach, and that of others, which relied upon a surface height function description, $h(x, t)$ [9]. There, the free-surface movement was purely radial.

The present problem frame of reference lies across the space-time domain for which the dynamics may be described through either an Eulerian or a Lagrangian frame-work. The generalised Arbitrary Lagrangian/Eulerian formulation (ALE) is a powerful technique that unites features of both such descriptions appealing to the concept of a *mesh velocity*. Typically, a material time derivative of a physical quantity, ϕ , within the reference configuration may be expressed in the form,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u}_c \cdot \nabla \phi$$

$$\mathbf{u}_c = \mathbf{u} - \mathbf{u}_m$$

with convective velocity, $\mathbf{u}_c$, fluid velocity, $\mathbf{u}$, and mesh velocity, $\mathbf{u}_m$. The particular form of the Arbitrary Lagrangian/Eulerian formulation employed relies upon the allocation of mesh velocity: being purely Eulerian if $\mathbf{u}_m=0$; purely Lagrangian if $\mathbf{u}_m=\mathbf{u}$; hybrid configurations covering alternative choices in between. In the present study, we adopt an *ALE* interpretation, for which $\mathbf{u}_m=\mathbf{u}$, and $\mathbf{u}_c=0$ on all boundaries. On interior nodes, $\mathbf{u}_m$ is taken as approximated by pure axial extension (centreline $\mathbf{u}_z$). This avoids the complication of mesh-folding, that may occur otherwise in the vicinity of the plates. Corresponding convective terms in momentum and constitutive equations are discarded at the continuous level. At the discrete level and for temporal accuracy, we have incorporated an additional convective correction step within the fractional-stages over each time-step. Nevertheless, practical experience has revealed that this correction incurs negligible contribution to such an extensionally-dominated flow and may be suppressed accordingly.

3.2 Discretisation of field equations

To solve the relevant governing field equations, we employ a time-dependent hybrid finite volume/element discretisation. This scheme is articulated in the form of a high-order, semi-implicit, incremental pressure-correction algorithm (TGPC) [25], employing a two-step Lax-Wendroff approach, and a Taylor series expansion up to second-order in time. A three-stage system emerges, constructed through operator splitting in space and time, forming a reduced fractional-staged set of equations. A non-solenoidal velocity field is computed at stage one via a predictor-corrector scheme. At stage two, temporal pressure-difference on the time-step is computed via a Poisson equation. Finally, a solenoidal velocity field is recovered at stage three to complete the time-step loop, see Refs. [25, 26]. This particular numerical algorithm is further modified to enhance temporal quality via an area-weighting procedure (α_l^T - *lhs* version, see [9] for practical implementation detail).

The resulting generalised space-time discrete matrix-vector system may be expressed as,

$$\begin{aligned}
\text{Stage 1a:} \quad & A_u(\mathbf{U}^{n+\frac{1}{2}} - \mathbf{U}^n) = b_u^a(p^n, p^{n-1}, \mathbf{U}^n, \mathbf{T}^n, \mathbf{D}^n), \\
& \frac{2}{\Delta t} A_i^\tau(\mathbf{T}_i^{n+\frac{1}{2}} - \mathbf{T}_i^n) = b_\tau(\mathbf{U}^n, \mathbf{T}_i^n, \mathbf{D}^n), \\
\text{Stage 1b:} \quad & A_u(\mathbf{U}^* - \mathbf{U}^n) = b_u^b(p^n, p^{n-1}, \mathbf{U}^n, \mathbf{U}^{n+\frac{1}{2}}, \mathbf{T}^{n+\frac{1}{2}}, \mathbf{D}^{n+\frac{1}{2}}), \\
& \frac{1}{\Delta t} A_i^\tau(\mathbf{T}_i^{n+1} - \mathbf{T}_i^n) = b_\tau(\mathbf{U}^{n+\frac{1}{2}}, \mathbf{T}_i^{n+\frac{1}{2}}, \mathbf{D}^{n+\frac{1}{2}}), \\
\text{Stage 2:} \quad & A_2(p^{n+1} - p^n) = b_2(\mathbf{U}^*), \\
\text{Stage 3:} \quad & A_3(\mathbf{U}^{n+1} - \mathbf{U}^*) = b_3(p^n, p^{n+1}), \tag{9}
\end{aligned}$$

where

$$A_u = \frac{2Re}{\Delta t} M + \frac{\mu_2}{2} S,$$

$$\begin{aligned}
b_u^a &= \{F_g - [\mu_2 S + Re N(U)]U - B_1 T_1\}^n + L^T \{P^n + \theta_1(P^n - P^{n-1})\}, \\
b_u^b &= \{F_g - [\mu_2 S + Re N(U)]U - B_1 T_1\}^{n+1/2} + L^T \{P^n + \theta_1(P^n - P^{n-1})\}, \\
F_g &= \{0, Bo/Ca\}^\dagger, \quad 0 \leq \theta_1 \leq 1, \\
A_2 &= K, \quad b_2 = \frac{2Re}{\Delta t} L U^*, \quad A_3 = \frac{2Re}{\Delta t} M, \quad \text{and} \quad b_3 = -L^T (P^{n+1} - P^n).
\end{aligned}$$

Matrix-operator notation above employs M , S , and $N(U)$ to represent the consistent mass-matrix, diffusion matrix and advection matrix, respectively. We note with the *ALE*-formulation used that convection terms vanish. Over the problem domain Ω , corresponding matrices may be expressed as,

$$M_{ij} = \int_{\Omega} \phi_i \phi_j d\Omega, \quad S_{ij} = \int_{\Omega} \nabla \phi_i \nabla \phi_j d\Omega, \quad N(U)_{ij} = \int_{\Omega} \phi_i (\phi_l U_l) \nabla \phi_j d\Omega. \quad (10)$$

The pressure stiffness matrix, K , and remaining matrices adopt forms:

$$K_{ij} = \int_{\Omega} \nabla \psi_i \nabla \psi_j d\Omega, \quad (L_k)_{ij} = \int_{\Omega} \psi_i \frac{\partial \phi_j}{\partial x_k} d\Omega, \quad (B_1)_{ij} = \int_{\Omega} \nabla \phi_i \phi_j d\Omega. \quad (11)$$

The additional stress-related matrices and vectors (A_i^τ and b_τ) are described below under fv-context. In the momentum equation, diffusion terms are approximated with a Crank-Nicolson discretisation over each time-step, introducing implicitness and stability to the scheme. Velocity on the parent triangular *fe*-cell is represented through quadratic interpolation ($\phi_j(\mathbf{x})$), whilst that on pressure is linear ($\psi_j(\mathbf{x})$). Such trial space interpolation may be represented in nodal (j) notation, viz.

$$u(\mathbf{x}, t^n) = \sum_j \phi_j(\mathbf{x}) U_j^n, \quad p(\mathbf{x}, t^n) = \sum_j \psi_j(\mathbf{x}) P_j^n. \quad (12)$$

An accelerated Jacobi iteration is employed to solve the resulting algebraic equations at stages one and three, where no explicit assembly of system matrices is required. It demands only a few iterations. This computationally efficient element-by-element procedure is space optimal, reflecting overall linear space-complexity in nodal degrees of freedom, and likewise linear time-complexity per time-step. The Jacobi scheme is an appropriate choice, being well-suited for parallelisation (see Grant et al. [27]) and for these mass-matrix equations, according to their suitable conditioning properties. In contrast, a direct Choleski reduction scheme is employed at stage two, to compute current pressure-difference over the time-step. The incremental version of this same scheme draws upon back-reference at stage one to the previous time-step pressure-difference. This yields superior uniform temporal error bounds, over the alternative non-incremental pressure-correction schema. The choice of direct solver for the scalar pressure-difference equation upholds the order of efficiency of the overall scheme and avoids iterative solution at the second-stage. Here, conditioning often provokes severe practical difficulties and is best handled independently of the remainder of the system. The symmetrical, sparse, banded nature of the resultant matrix involved renders computational efficiency attainable, even for three-dimensional implementations, see Grant et al. [27].

3.3 Sub-cell finite volume stress discretisation

Cell-vertex *fv*-schemes applied to stress are based upon an upwinding technique (fluctuation distribution), that distributes control volume residuals to provide nodal solution updates. Briefly, by rewriting the stress constitutive equation (3) through flux ($\mathbf{R}$) and source ($\mathbf{Q}$), one may obtain:

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = -\mathbf{R} + \mathbf{Q}, \quad (13)$$

$$\mathbf{R} = \mathbf{u} \cdot \nabla \boldsymbol{\tau}, \quad (14)$$

$$\mathbf{Q} = \frac{1}{We} (2 \frac{\mu_1}{\mu} d - \boldsymbol{\tau}) + L \boldsymbol{\tau} + \boldsymbol{\tau} L^T. \quad (15)$$

We consider each scalar stress component, τ , acting on an arbitrary volume Ω , whose variation is controlled through the fluctuation of components of the flux and source terms, R and Q respectively,

$$\frac{\partial}{\partial t} \int_{\Omega} \tau d\Omega = - \int_{\Omega} R d\Omega + \int_{\Omega} Q d\Omega. \quad (16)$$

The goal is to evaluate these flux and source variations on each finite volume triangle, and to distribute the same to its three cell vertices, according to the preferred strategy. The update to a given node l is obtained by summing contributions from its control volume Ω_l , which is composed of all *fv*-triangles surrounding node l , see [9]. Flux and source residuals may be evaluated over different control volumes. Namely, over the *fv*-triangle $T(R_T, Q_T)$ and/or the median dual cell (*mdc*), associated with a given node l within the *fv*-cell $T(R_{mdc}, Q_{mdc})$. Median-dual-cell zones are non-overlapping regions, defined per node l , an area one-third of the triangular cell over which it is constructed, see [9].

Taking eq. (9) into account, a consistent distribution approach, devised by Webster et al. [28], may be expressed in the generalised form,

$$\{\Omega_{FD} + \Omega_{mdc}\} \frac{\Delta \tau_l^{n+1}}{\Delta t} = \underbrace{\sum_{\forall T_l} \delta_T \alpha_l^T b_{T_l}}_{FD} + \underbrace{\sum_{\forall mdc_l} \delta_{mdc}^T \alpha_{mdc}^T b_{mdc}^l}_{mdc} \quad (17)$$

where

$$b_{T_l} = (R_T + Q_T), \quad b_{mdc}^l = (R_{mdc} + Q_{mdc})^l$$

$$\Omega_{FD} = \sum_{T_l} \delta_T \alpha_l^T \Omega_{T_l}, \text{ and } \Omega_{mdc} = \sum_{mdc_l} \delta_{mdc}^T \Omega_{mdc_l}^T.$$

The parameters, δ_T and δ_{mdc}^T are mutually linked and complementarily exclusive, $\delta_{mdc}^T = 1 - \delta_T$, and may be established to provide a powerful categorization of this class of schemes, see [28, 29] for schemes CT_i, i=0-3. For example, $\delta_T = 1$ and $\delta_{mdc} = 0$ reverts to a pure fluctuation distribution form, pertinent to present extension-dominated needs. This scheme is temporally modified on the *lhs*, using α_l^T area-weighting.

Fluctuation distribution coefficients α_l^T , may be provided via an appropriate choice of scheme: for example, LDB, Lax-Wendroff, or PSI. Here, we focus on a single linear version (Lax-Wendroff, termed Lax), satisfying the property of linearity-preservation*. It is spatially-centred and second-order accurate in space. In addition, it contains a dissipation term, designed to control oscillations in the neighbourhood of discontinuities, thus conferring second-order accuracy in time. For this Lax-scheme, the distribution coefficients α_l^T may be expressed as:

$$\alpha_l^T = \frac{1}{3} + \frac{\Delta t}{4\Omega_T} \mathbf{a} \cdot \mathbf{n}_l^T, \quad (18)$$

where, Δt is a time-step size, $\mathbf{a}$ an averaged cell-advection velocity, and $\mathbf{n}_l^T$ a scaled inward-pointing normal vector to an edge of triangle T , opposing node l . Importantly, the closer the advection velocity $\mathbf{a}$ is to being parallel to a cell boundary, the larger the contribution to the downstream node at that boundary.

4. Problem specification

In this paper, we consider the extensional deformation of a viscoelastic filament between two coaxial discs, drawn apart in time at an exponential rate. The context is one where, in the first instance, it is reasonable to ignore the effects of inertia and gravity. Also, we limit ourselves to one initial aspect ratio of $\Lambda_0 = 0.54$, following Yao et al. [3]. The material and operating parameters of Table 1,2 were selected to accommodate single-mode models (Oldroyd, LPTT and Giesekus), and multi-mode Giesekus, LPTT models.

The boundary conditions and geometric domain considered are illustrated in Fig. 1. Dirichlet-type boundary conditions are imposed on all but the free-surface. In addition, initial conditions are taken as quiescent, with the exception of those on the moving-plates, where the initial impulsive velocity is taken as $(V_0 = L_0 \dot{\epsilon}_0/2)$. This provides an appropriate characteristic velocity scale, V_0 , with a length scale of the initial filament length, L_0 . Primarily, we consider single-mode predictions, but also present some multi-mode preliminary solutions for two Hencky-strain levels of $\epsilon = 0.2$ and $\epsilon = 1.0$. Hence, we are able to analyse the consequence of drawing upon multi-mode physical representation, as implemented for the first time via our proposed *hy-fV* scheme.

A typical rectangular initial mesh of 20×100 elements, with 8241 nodes and 4000 triangular elements, is used to perform present computations. This mesh choice was adopted after conducting an intensive mesh refinement study, the details of which are reported in [30]. The mesh must adjust in time, being stretched and distorted, its movement being controlled by the time-step and remeshing algorithm, see [9, 31].

The normal force (F_z) exerted at the moving end-plates is calculated through the integral of stress over the end-plate area A , viz.,

$$F_z(t) = \int_A (T_{zz} + p) dA, \quad (19)$$

where $T_{zz} = 2\mu_s D_{zz} + \tau_{zz}$ represents the total stress, summing viscous and polymeric components. The Trouton ratio Tr is defined by

* But not positivity, therefore.

$$Tr(t) = \frac{\bar{\mu}^+}{\mu_0}, \quad (20)$$

where $\bar{\mu}^+$ represents the extensional viscosity (type IB, as referred to by Yao et al. [13]), which enables us to compare the Trouton ratio with corresponding experimental data. The extensional viscosity is functionally specified as,

$$\bar{\mu}^+(t) = \frac{\dot{F}_z}{\dot{E} \pi R_{mid}^2} - \frac{\dot{\chi}}{\dot{E} R_{mid}} + O(F_i, F_g).$$

Here, the notation employed implies an extension rate ($\dot{E}$), surface tension coefficient (χ), and radius at the filament mid-plane (R_{mid}). The term $O(F_i, F_g)$ represents correction arising from imposed inertial (F_i) and gravitational forces and (F_g), respectively. This term may be neglected in the present study, enabling us to compare our results directly with those of Yao et al. [13].

4.1 Fluid characterization

In the present article, we consider data for a 5.0 wt% solution of narrow distribution, high molecular weight polystyrene, following the analysis of Yao et al. [3]. The viscometric characterization for this test fluid is described in Li et al. [2]. In Fig. 2, we plot the measured steady-state shear viscosity compared to that concomitant with the Giesekus model: covering instances of single, two and three-mode representations. We observe that the single-mode model predicts the experimental shear viscosity well up to a shear-rate of 10s^{-1} . Subsequently, due to the large contribution of the solvent viscosity, the shear viscosity deviates to a plateau level of μ_s (see on). Such deficiency in the single-mode model may be corrected by appealing to a two or three-mode approximation. Parameter values for the Giesekus model are provided in Table 1, extracted from Li et al. [2], and used by Yao et al. in the filament stretching context [3]. The single-mode model is identified by adopting the largest of the relaxation time modes, with the corresponding μ_s absorbing the partial-viscosities of the discarded modes.

The single-mode prediction is well represented in shear, up to a shear-rate of 10s^{-1} (as arise in the range of filament stretching deformation rates). Yet, one observes that this model will effectively ignore the shorter relaxation times response. This in turn will lead to the generation of rapid growth in transient stress. In contrast, the two- and three-mode models will accommodate a broader range of time-scales, being exposed to lower relaxation time response.

We note the comments of Li et al. [2] on rheological model suitability for present purposes, covering Giesekus, PTT and PTT-WM. In shear, the 3-mode Giesekus model was observed to provide the best representation for both flow inception and steady second-normal stress difference. This is mitigated by its inability to independently fit extensional data, being found to under-predict uniaxial extensional viscosity for shear thinning solutions. In contrast, the PTT model offers the ability to adjust extensional viscosity predictions over a wide range of deformation rates, yet suffers from the well-known drawback of the Gordon-Showalter derivative in shear flow. The preferred choice of Li et al. was a PTT-WM model, matching shear data well and being adjustable for extensional data, notwithstanding the fact that this model may prove suspect in transient flow. Implementation revealed that the PTT-WM model was certainly more computationally efficient than a multi-mode alternative. The present study casts further light on both Giesekus and LPTT predictions.

5. Results and Discussion

We proceed to outline our computational results for the weakly strain-hardening polystyrene solution data described above. From the shear viscometric material properties of the test fluid, it is clear that the single-mode fit provides a reasonable approximation over a reduced range of shear rates that is sufficient for present needs. We proceed to present comparative numerical predictions for Oldroyd, Giesekus and LPTT models, primarily based on the single-mode representation. Subsequently, we discuss the implications when a multi-mode representation is introduced. That is with the knowledge that a three-mode in contrast to a single-mode Giesekus model, provides an improved fit over a wider range of shear-rates.

5.1 Single-mode predictions

The principal comparison basis throughout this work is taken against the data of Yao et al. [13] who used two Deborah numbers (De) of 1.97 and 1.89. We adopt $De = 1.89$ though draw upon comparison against both cases used by Yao et al. To suit present purposes, we choose to adopt a different length scale from Yao et al., ours taken as the initial filament length, theirs being the radius. We refer throughout to critical levels of Hencky-strain achieved (ϵ_{crit}) prior to observing numerical failure in the stretching procedure.

Early Hencky-strain predictions: spatial comparison ($\epsilon=1.0$)

We begin with a comparative review of Oldroyd, Giesekus and LPTT fluid kinematic/stress results, covering spatial plots at a specific Hencky-strain of unity ($z=0$). We comment that as the mobility factor α tends to zero, we recover the structure of the Oldroyd-B model. By examining the kinematics along the mid-plane and centreline axes of the filament, we observe that the most important differences in results between the three models reside near the free-surface zone. Through the radial velocity V_r along the mid-plane in Fig. 3a), we observe that velocity profiles depart around a radial value of about $r=0.1$ units. This has a significant impact on stress development, as we see below. V_r -maxima for the Giesekus model are larger than those corresponding to LPTT and the Oldroyd-B model (0.65 compared to 0.60, and 0.55 units). This position leads to more rapid compression towards the centreline under the Giesekus than with LPTT and Oldroyd-B models. In contrast, along the centreline axis, where the axial velocity V_z is dominant, we do not detect any major differences in profiles associated with these two models (see Fig. 3b).

In Fig. 3c), we plot the stress component τ_{zz} along the mid-plane of the filament. Here, τ_{zz} is elevated in value, from 3.5 units for the Giesekus model to about 4.0 units for LPTT, and 5.5 units for the Oldroyd-B model. In Fig. 3d), we plot the stress τ_{zz} along the centreline of the filament. Again, trends are similar for the three models. Stress is minimal near the end-plates, increasing towards a maximum at the mid-plane of the filament, achieving values of 4.5 units for Oldroyd-B, 3.5 units for LPTT and 3.0 units for Giesekus instances. These findings are in keeping with expectation due to the differences in elongational behaviour for these respective models.

Once more, along the axial direction and for velocity gradient, D_{zz} , we observe greater departure between results for Oldroyd-B, LPTT and Giesekus models near the centre of the filament (see Fig. 3f). The more excessive flattening of the Oldroyd D_{zz} -profile goes alongside the tendency towards more uniformity in filament column radius and greater

propensity to strain-harden. Along the radial direction at the filament mid-plane, the Giesekus model generates larger D_{zz} -values than with LPTT and Oldroyd-B models (see Fig. 3e). For all three models, D_{zz} sustains a constant level (about 1.2 units for Oldroyd, 1.3 for LPTT, and 1.4 for Giesekus) over the range $r/R_{min} < 0.5$. Then, D_{zz} increases to reach a value of about 1.35 units for Oldroyd, 1.45 for LPTT, and 1.6 units for Giesekus at the free-surface. The rationale for this is that at the filament mid-plane, axial velocity change between filament layers is faster for the Giesekus than the Oldroyd-B model. This implies that in the Giesekus representation, particles at the mid-plane (particularly near the free-surface) move faster in the axial direction (z). Hence, the filament is more easily extended in the case of the Giesekus model fluid than equivalently for the Oldroyd-B form.

Temporal predictions at larger Hencky-strains ($0.2 \leq \epsilon \leq 3.0$)

In Fig. 4, we consider the development of the mid-plane filament radius (R_{mid}) as a function of Hencky-strain. One observes the anticipated overall trend of radial decrease with filament stretch. For the Giesekus model, radial evolution overlaps that for approximate lubrication theory ($e^{-0.75\epsilon}$ curve, Newtonian, fixed-end-plates result) up to a Hencky-strain of two. Beyond this level of $\epsilon=2$, lubrication theory overestimates R_{mid} , with the true mid-plane radius decreasing more sharply, following a non-linear-logarithmic representation. In contrast, R_{mid} for the Oldroyd-B model departs from lubrication theory around $\epsilon=1.2$ ($e^{-0.75\epsilon}$ curve), retarding the rate thereafter, approaching the curve representative of homogeneous deformation in uniaxial elongational flow ($e^{-0.5\epsilon}$ curve). Similar findings were reported by Yao et al. [13]. In R_{mid} , we observe that the LPTT model response is closer to that for Oldroyd-B than Giesekus, up to a Hencky-strain of three, being captured between these two extremes. There is slight reduction noted in filament radius from Oldroyd to LPTT results. Beyond $\epsilon=3$, the pattern begins to dramatically adjust, so that R_{mid} begins to increase (reversal of decreasing trend) with local minima appearing off the centreline, a behaviour unobserved with other constitutive models. This may be attributed to the larger transient elongational viscosity response encountered with the LPTT model. The consequence of this increase in mid-plane radius, alongside a highly localised adjustment in filament shape thereabouts, lies in the formation of a bead-like structure in the filament.

To analyse the flow kinematics for the three models of Oldroyd-B, Giesekus and LPTT, we display in Fig. 5 axial profiles for velocity gradient D_{zz} along the centreline of the filament. We gather that there is barely any difference across these models at lower reaches of Hencky-strain up to $\epsilon=0.2$. The overall trend is that the axial velocity gradient increases from a vanishing value at the end-plates to a maximum at the filament mid-plane, where the largest stretching occurs. For $\epsilon \geq 1$ and in accordance with the literature, the Oldroyd-B model displays flatter centralised profiles with less extension at the mid-plane than arises with LPTT and Giesekus forms. In contrast, larger extension is encountered with Oldroyd-B results near the end-plates. At the elevated Hencky-strain of $\epsilon=3$, the solution for the LPTT model begins to show non-symmetric structure, and D_{zz} -maxima shift away from the mid-plane to a location in the upper half of the filament. This response may be due to the competition between opposing influences of shear-thinning and extreme strain-hardening, typical of force imbalance, commonly encountered between inertial and surface tension effects, see below. In contrast, the less hardening Giesekus results do not lose their symmetrical form, with D_{zz} -maxima located at the filament mid-plane once again.

To interpret the position more clearly, we look to the axial velocity V_z , itself taken along the filament axial-centreline. The overall pattern in V_z of Fig. 6 displays minima at the

bottom-plate and maxima at the top-plate, vanishing at the filament mid-plane. Discrepancy between models commences around Hencky-strains of unity. The difference between the Giesekus result and that for the two alternative models reveals itself as the filament is stretched further, whilst at this stage maintaining the symmetrical behaviour across models. However at Hencky-strains of $\epsilon \geq 3$ the symmetry is disturbed with the LPTT model, and the axial velocity vanishes in the upper-portion of the filament instead of at the mid-plane. This does not arise in the case of the Giesekus model with the filament maintaining symmetry throughout the stretching period.

As a consequence, we turn to interrogate the development of the τ_{zz} axial stress component along the filament axial-centreline for different Hencky-strain steps. Here, we illustrate in Fig. 7(a-d) snapshots for $0.2 \leq \epsilon \leq 3.0$; with $\epsilon_{\text{crit}}=2.6$ for Oldroyd-B results, whilst LPTT survives up to $\epsilon_{\text{crit}}=3.6$ and Giesekus up to $\epsilon_{\text{crit}}=4.0$. We rationalise these findings through the extremes of stress and deformation observed in each case. Overall, all models furnish similar trends up to $\epsilon=2.6$. The axial stress relaxes at the end-plates, increasing in value towards the filament mid-plane. The difference lies in τ_{zz} -maxima reached for each model. Beyond a Hencky-strain level of $\epsilon=0.2$, Oldroyd-B solutions yield the largest axial stress, followed by those for LPTT, and finally those for the Giesekus model. This we may attribute unequivocally to the transient elongational viscosity properties for these three alternative models. In comparison to Giesekus, the LPTT model begins to show more bizarre flow structure and stress response at extreme levels of Hencky-strains, $\epsilon \geq 3$. For the LPTT model, maxima in stress gradually shift into the filament upper-portion. Then, axial stress decreases at the mid-plane, to increase thereafter in the filament lower-portion, before relaxing at the bottom-plate. This trend is replicated and amplified at still larger Hencky-strain levels up to $\epsilon=3.6$. τ_{zz} -maxima range from 0.3 units at $\epsilon=0.2$, to 100 units at $\epsilon=3.6$ (see Fig. 7(e-f)). Correspondingly for the Giesekus model, τ_{zz} -maxima range from 0.3 to only 15 units. One may observe that these values are much lower than for the LPTT choice, due to the large differences in transient elongational properties between these two model alternatives. As the Oldroyd-B model displays unbounded extensional dynamics, the equivalent problem is considerably more severe to solve when compared to those for the LPTT or Giesekus models. This is demonstrated through ϵ_{crit} levels reached. So for example, for this particular aspect ratio and solvent:polymeric ratio, and when employing an identical Hencky-strain step ($\Delta t_{\text{Hencky}}=10^{-3}$), we have been able to reach $\epsilon_{\text{crit}}=2.6$ only with Oldroyd-B. A smaller Hencky-strain step would be necessary to stretch the filament somewhat further.

Temporal variation in stress-maxima, τ_{zz} and τ_{rz} , for Oldroyd, Giesekus and LPTT are demonstrated in Fig. 8a, and Fig. 8(b-c) illustrates the variation of deformation rates (in shear and extension) with strain (time). Stress fields and profiles indicate that the extensional component of polymeric stress τ_{zz} rises exponentially with the strain level. At any particular strain level, say for example $\epsilon=2.5$, Oldroyd-B being the most strain hardening fluid provides the largest value of stress (80 units) units, compared to LPTT (20 units) and followed by Giesekus results (10 units). The shear stress component, τ_{rz} , is much smaller in comparison to its extensional counterpart. At the Hencky strain level of 2.6, τ_{rz} for Oldroyd-B is about 6% of τ_{zz} . Deformation rates D_{zz} and D_{rz} display similar response patterns to each other. For the Oldroyd-B fluid, D_{rz} (3 units) is larger when compared to D_{zz} (1.3 units) at a Hencky-strain of 2.0. In contrast, D_{rz} for the Giesekus model drops to small values at $\epsilon=2.0$ (around 0.2 units).

In Fig. 9, we compare Trouton ratio predictions plotted against effective Hencky-strain by common convention and against the available experimental data of Yao et al. [13]. The numerical results agree well with the experimental data at low Hencky-strains of $O(\text{unity})$;

this being continued likewise for the Giesekus model at larger Hencky-strain levels. The departure between Giesekus and LPTT-predictions is observed around $\epsilon=1.2$, beyond which stage the LPTT-Trouton ratio increases at a greater rate than that observed for the Giesekus model. This is in accordance with the dynamic extensional properties of each.

To interrogate filament structure globally and within the zone of the filament foot, we plot in Fig. 10 filament shape for the different models at $\epsilon=2.6$. Near the end-plates and in the filament-foot zone, Oldroyd-B filament shape is more pinched toward the axial-centreline of the filament than is apparent for LPTT and Giesekus instances. In contrast, the Giesekus model filament-foot limits lie outside the others, so that the fluid is less compressed towards the axial-centreline than for the LPTT instance. The same trend was reported by Yao et al. [13] for Oldroyd-B and Giesekus models. In contrast at the filament mid-plane, the Oldroyd-B fluid is less compressed towards the centreline than with the LPTT-fluid, which in turn is less compressed than with the Giesekus fluid. This may be attributed once more to the nature and differences between the elongational properties of these three fluids. That is, the larger the elongational viscosity, the more difficult it becomes to compress the fluid towards its mid-plane/centreline intersection point. This is the region of largest extension where the fluid will resist compression most.

Quarter-domain versus half-domain problem (Oldroyd-B and LPTT)

In order to verify mesh convergence and to validate non-symmetrical results, we have employed a quarter-domain problem alongside the default half-domain case (full-length filament). Here, only a quarter of the filament domain is physically retained and solved for in a symmetric manner, effectively doubling the mesh-density of the original problem. In this case, symmetry boundary conditions with no-shear must be explicitly imposed on the filament mid-plane (now lower boundary). Such a setting effectively precludes the development of asymmetrical features in the evolving flow and is useful as such in comparison against the full-length filament representation.

In Fig. 11, we start by plotting minimum filament radius at the mid-plane axis (R_{mid}) and filament-centre stress against increasing Hencky-strain for Oldroyd-B and LPTT models. Results are presented for Oldroyd and LPTT instances, as the former provides premature $\epsilon_{crit} < 3.0$, whilst the latter transcends $\epsilon=3.0$ and displays bead-structures with localised minima. The minimum radius in Fig. 11a,b for the quarter-domain problem overlaps that corresponding to the full-length case, exactly for Oldroyd-B, with slight departure noted in LPTT around $\epsilon \approx 3.0$. According to the experience of Yao et al. [13], the radius should continually decrease as the filament is stretched up to $\epsilon=4$, in accordance with our full-length problem Giesekus results. Similar agreement is detected in Fig. 11c,d for the development of τ_{zz} across fluids and the two problem-domain instances. Stress increases monotonically to a maximum for the Oldroyd-B model ($\epsilon_{crit}=2.6$ full-length case, $\epsilon_{crit}=2.8$ quarter-domain problem); for LPTT fluids, τ_{zz} increases to a maximum, beyond which there is decline thereafter ($\epsilon_{crit}=3.6$ full-length case, $\epsilon_{crit}=3.4$ quarter-domain problem). Such findings confirm trends observed with mesh refinement.

Quarter-domain solutions consistently report differing ϵ_{crit} than do their full-length counterparts. The Oldroyd position would indicate that increased refinement has aided in elevating ϵ_{crit} . Under LPTT, reduction in ϵ_{crit} is possibly due to anticipation of asymmetry looming (or see on). The marginal solution discrepancies noted are subtle in that specific local conditions about the symmetry mid-plane and their discrete treatment come into play here. Our detailed investigation on this issue has revealed that reflected meshing about the filament

mid-plane (to impose symmetry conditions numerically), physically retained, or virtually imposed (as with the quarter-domain problem), itself may give rise to some numerical anomalies. The issue here emerges through solution (velocity-)gradient representation across such a flow station, and when treated as a domain boundary or not, as the case may be. Asymmetrical, continuous structured meshing across the mid-plane, as we have chosen to employ elsewhere by default, retains the discrete property of superconvergence [32]. This is reflected in the computational stencils invoked and the high quality of extracted velocity gradients thereby. The same cannot be said for reflected meshing, which may demonstrate false local minima in normal gradients (producing a reflected oscillation in D_{zz}) on passing normally across the mid-plane. One might argue that such an anomaly is minor in extent and natural for such a domain periphery location, where degradation in accuracy is to be anticipated in the quarter-domain problem¹. Certainly, its significance is heightened along a purely extensional symmetry flow line, when steep velocity gradients are anticipated. For example, simply capping the maxima in D_{zz} with a plateau across the mid-plane would correspond to a crude finite-difference stencil approximation. This would effectively avoid an undershoot in D_{zz} but fail to capture the true maxima accurately. Nevertheless, in the context of seeking representation for physical disturbances and non-symmetrical solutions, this is a point of detail to recognise, which has influenced our preference towards the full-length problem representation with the superconvergent choice of continuous structured meshing, non-reflective across the filament mid-plane.

5.2 Inclusion of gravity and surface tension (LPTT)

We chart the influence of inclusion of both surface tension and axial gravity in separate phases, noting results for LPTT by default, against which other fluids may be compared below. We begin with low Hencky-strain predictions.

Surface tension effects, $\epsilon \leq 2.0$

First, with *surface tension alone*, quantified through the inverse Capillary Number, denoted by Ca^{-1} , covering a range of values from 0.0, 0.1, 0.5 and 1.0 (nb. Yao et al. [13] employed $Ca^{-1}=0.1$). Trends in τ_{zz} demonstrate for $\epsilon=1.0$ (see Fig. 12b), the localised maxima that develop at the position of minimum radius, whilst filament shape remains symmetrical. At $Ca^{-1}=1.0$, local stress magnitude can be exaggerated by as much as 10% above the equivalent scenario devoid of surface tension influence. These local maxima are absent at the earlier time of $\epsilon=0.2$ (see Fig. 12a). At larger levels of Hencky-strain of 2.0 (see Fig. 12c) and above, we can observe that solutions in τ_{zz} , begin to distort for Capillary numbers in excess of $Ca^{-1}=0.5$, this becoming quite marked by the level of $Ca^{-1}=1.0$. These findings demonstrate the sensitive nature of the solution to this particular physical effect. Correspondingly, filament shape is barely affected up to Hencky-strains of 2.0 (see Fig. 12d).

Inertial and surface tension influence combined, $\epsilon \leq 2.0$

At $\epsilon=1.0$ and when *inertial-gravitational forces* (imposed axially) are included alongside surface tension effects, $0.1 \leq Ca^{-1} \leq 1.0$, with a base level of body force factor of $F_g=0.317$, there is distortion in filament shape and surface tensile stress distribution. This distortion is shifted

¹ We believe this to be a feature often overlooked in some flow problems possessing symmetry boundaries, e.g. cross-slot flow or flow past a sphere.

towards the lower-half portion of the filament, where the minima in filament radius and local stress maxima now occur (see Fig. 13). We observe this position once more at the more elevated Hencky-strains of $\epsilon=1.6$ and 2.0 . By removing surface tension here, we can detect that surface tension forces (at $Ca^{-1}=0.1$ level) adopt a restoring role, acting to counterbalance against gravitational forces. As such, local maxima in stress are shifted backwards towards the filament centre.

Inertial influence alone, $\epsilon \leq 2.0$

One may chart (see Fig. 14) the effect of inertial body force without surface tension through progressive Hencky-strains to observe the gradual build-up in magnitude and shift for this asymmetrical feature. Further amplification of body force effects, via increase in F_g to 0.737 , promotes the slump even more towards the lower-half of the filament. So at $\epsilon=2.0$, maxima in stress increase by some 4% from $F_g=0.317$ above the $F_g=0.0$ state, to a further 12% measured from $F_g=0.737$ above the $F_g=0.317$ state. There is continual and gradual departure even further away from the filament centre as F_g increases. As above, inclusion of surface tension barely affects these levels but acts somewhat to restore symmetry in stress-profile shapes.

Large Hencky strains results, $\epsilon > 2.0$

At large Hencky strains above 2.0 , various conflicting influences are brought to bear, so that we observe a richness in alternative flow structures generated, dependent upon the particular configuration involved. This includes asymmetries in shape and the formation of bead-like structures, alongside redistribution and replication of local maxima in stress. We comment on filament shape, via profiles and full filament representations, and on tensile stress (τ_{zz}) and/or axial deformation rate (D_{zz}) likewise. For the characteristic instance involving *surface tension* (at $Ca^{-1}=0.1$) and *gravity* (of $F_g=0.317$), see Fig. 15a for LPTT-fluid, we find that filament shape becomes disrupted from its symmetrical orientation above $\epsilon=2.8$, so that at $\epsilon=3.0$ there is a slight indentation around the filament centre. This trend develops further up to the critical levels of $\epsilon_{crit}=3.2$ with filament mid-plane τ_{zz} -maxima of 112 units, producing two small bulges (beads) on either side of the filament mid-plane. Counterpart stress patterns, as displayed in Fig. 15b, indicate that there are local extrema in stress corresponding to these bulges, of relative size some 16% of the mid-plane maxima. For Hencky-strain values of $\{2.0, 2.4, 2.6, 2.8, 3.0, 3.2\}$, the corresponding mid-plane τ_{zz} -maxima vary through values of $\{16.7, 24.2, 28.4, 37.3, 55.0, 112.3\}$. In addition, we are able to extract more information here when tracing the temporal evolution of these pointwise maxima, in stress $\{\tau_{zz}$ and τ_{rz} (Fig. 16a) } and deformation rates $\{D_{zz}$ and D_{rz} (Fig. 16b) }. This illustrates dominance between extensional and shear components of stress within the system. Comparing this information against that presented earlier without surface tension and inertia (see Fig. 8), it is apparent that the stress and deformation rate maxima patterns follow similar behaviour in both scenarios (stress increases and deformation rates decrease with time).

In switching attention to the instance without surface tension, so subject to *gravitational forces alone*, we are now able to press on further in critical Hencky strain up to $\epsilon_{crit}=3.4$. In Fig. 17a, with results provided above $\epsilon=3.2$, we are able to make direct comparison against the previous configuration with surface tension. Now, only a single bulge-bead structure emerges and this is shifted away from the centre of the filament towards its lower-half. Here, there is less radial pinching about the bulge, so that it is more disperse than when surface

tension is present. The τ_{zz} -maxima of 93.4 units (see Fig. 17b) emerges in the lower filament portion with bead-extrema representing some 20% of this. For Hencky-strain values once more in the range $2.0 \leq \epsilon \leq 3.4$ the corresponding τ_{zz} -maxima vary through values of {15.8, 22.2, 25.4, 31.3, 45.0, 83.3, 93.4}.

Within the isolated scenario of *surface tension alone* yet without inertial body forces, critical Hencky strain returns to $\epsilon_{\text{crit}}=3.2$. So, this is common across all instances with surface tension. The asymmetrical response in bead-like structure and its build-up beyond $\epsilon=3.0$, switches to influence only the upper half-portion of the filament. This is in complete juxtaposition to that detected with inertia alone (see Fig. 18 (a-b)). The τ_{zz} -maxima of 81.1 units emerges slightly off-centre in the lower filament portion with bead-extrema representing around 16% of this. Corresponding τ_{zz} -maxima over Hencky-strain range $2.0 \leq \epsilon \leq 3.2$, now vary through values of {15.9, 22.6, 26.7, 31.8, 41.3, 81.1} - following closely those for the instance with inertia alone.

In contrast, the situation *without gravity and surface tension* (Fig. 19) provides symmetry and no bead-structure up to around $\epsilon=3.0$. The significant features are that τ_{zz} -maxima are lower here than in the above cases at $\epsilon=3.0$, only reaching values of 34.0 units. There are signs however that stress maxima are splitting about the central-plane beyond $\epsilon=2.8$, so that at $\epsilon=3.0$, the mid-plane value is some 26 units, being 25% lower than the actual off-plane maxima. In this scenario, we may pass comment that the Giesekus model realised a $\epsilon_{\text{crit}}=4.0$, with a τ_{zz} -maxima on the central-plane of 12 units, reflecting no bead-like or asymmetrical features (see Fig. 20).

5.3 Multi-mode predictions

In this section, we cast some light practically upon the departure detected between solutions gathered for single-mode and multi-mode models approximations, ignoring inertial and surface tension influences. Differences will arise due to the larger solvent viscosity incorporated into the single-mode case, and the wider spectrum of relaxation times represented in the multi-mode instance. The significance in impact of these differences on single-mode predictions is our concern. With the multi-mode representation, shorter response times will have some bearing upon the solutions generated. To demonstrate solution response for shorter times, we plot in Fig. 21 the predicted Trouton ratio, at the mid-plane given by Eq. 20. At the outset of the stretching process with the single-mode model, the Trouton ratio is dominated by the Newtonian solvent component, as the polymeric component is negligible. This leads to the generation of a larger initial Trouton ratio, to increase at a constant rate thereafter with further stretching. In contrast for a *two-mode* model, the viscous contribution is miniscule when compared to its polymeric counterpart. This switch in component dominance will be reflected through the total stress that governs the development of the axial force, and also influences the temporal evolution of the multi-mode Trouton ratio. Tr commences from a smaller initial value than for single-mode, and subsequently increases monotonically at an increased rate over the single-mode case. Similar observations were also gathered by Yao et al. [3].

Kinematic and stress solutions

In Fig. 22a, we plot the axial velocity component along the filament centreline ($r=0$) at a Hencky-strain level of $\epsilon=0.2$. At this level of Hencky-strain, there is barely any difference between the single and multi-mode velocity results. In contrast, the axial stress component τ_{zz}

shows signs of adjustment between the single- to multi-mode instances (see Fig. 22c-d). The general spatial trend in all τ_{zz} modes follows that of single-mode solutions, reaching its maxima at the filament mid-plane. As we can observe in Fig. 22c, centreline τ_{zz} is reduced in the single-mode case. In the absence of the shorter relaxation times, the total stress is dictated by the largest mode only. This in turn disregards the effect of shorter times, as we can gather from the increase in multi-mode τ_{zz} . Likewise, the radial stress component, τ_{rr} , reflects similar behaviour to that in τ_{zz} (see Fig. 22c, d); yet, τ_{rr} -mid-plane maxima are reduced in comparison to that in τ_{zz} . Velocity and stress show little discrepancy elsewhere throughout the domain, across single and multi-mode solutions. This is displayed in Fig. 23(a-c) for Giesekus, and Fig. 23(d-f) for LPTT, where small solution differences are principally located near the free-surface region.

When the filament is stretched further up to a Hencky-strain of $\varepsilon=1.0$ the departure between the single and multi-mode solutions become more apparent at the free-surface region (see Fig. 24). There, curvature adjustment is more extreme in the multi-mode form, being pushed further towards the centreline than with the single-mode representation.

At the small Hencky-strain of $\varepsilon=0.2$, and along the centreline axis of the filament, we observe that there is barely any difference in the axial extension rate, D_{zz} , between single- and multi-mode representations. This is true for both models, LPTT and Giesekus, as displayed in Fig. 25. In contrast, as the filament is stretched and the Hencky-strain increases, discrepancy in D_{zz} becomes more apparent. That is, between single- and multi-mode approximations, and between LPTT and Giesekus models. The multi-mode representation yields a slightly larger elongation rate than in the single-mode case about the filament mid-plane. Concerning the difference in response between LPTT and Giesekus models, the larger axial elongation rate, D_{zz} , corresponds to the LPTT model. This will in turn have an influence upon the axial stress component, τ_{zz} . Again, at a Hencky-strain of $\varepsilon=0.2$, the axial stress is comparable to that for LPTT and Giesekus fluids, with either single or multi-mode approximations (see Fig. 25). Here, the multi-mode representation leads to a slightly larger axial stress than is apparent with the single-mode alternative.

Overall in axial stress, the multi-mode representation generates deeper penetration across the filament span-wise than in the corresponding single-mode case, as displayed in Fig. 26 for the LPTT model. Larger stress values are observed near the mid-plane and free-surface region across the domain with the multi-mode representation at Hencky-strains of unity (10 units multi-mode, compared to 3.5 units single-mode). At the smaller Hencky-strain $\varepsilon=0.2$, such stress values were comparable. Generally, the multi-mode representation would be held to provide a better match to the material functions, throughout a wider range of deformation rates up to $O(10^3 \text{ s}^{-1})$. Nevertheless for the present problem, the single-mode approximation has proved itself to be reasonably adequate, picking out the salient features of the solution and particularly as shear-rates are relatively limited to $O(10 \text{ s}^{-1})$. This position and the computational efficiency of the single-mode implementation strongly commends its choice, as opposed to a multi-mode alternative.

6. Conclusions

This transient filament stretching study has advanced upon our earlier work in this area in dealing with Oldroyd fluids, to consider more realistic extensional flow response through Giesekus and LPTT models. Our ALE- h_y -fV scheme has proven more than capable in handling such a severe numerical challenge, capturing sharp differences in filament curvature and flow structure throughout flow evolution. A single-mode approximation has proven

adequate to represent the principal features of the flow, as they emerge and within the range of deformation rates of interest. Full filament and quarter-domain problem comparison has confirmed the quality of solutions generated. If anything, multi-mode approximation at larger Hencky-strains does point to deeper stress penetration across the filament span and reinforces single-mode predictions on Trouton ratio (so, extensional viscosity also).

At Hencky-strain levels greater than three, bead formation is observed with the more strain-hardening LPTT fluid, features that are absent in the equivalent Giesekus case. For the current choice of problem aspect and viscosity ratio, the Oldroyd-B fluid could not reach such high levels of Hencky-strain. The influence of gravitational inertia is to create a directional bias to the filament shape and to counterbalance surface tension effects. The inclusion of such forces has been analysed for the LPTT model, both with and without surface tension, to judge their relative and localised significance. Current findings lie in close agreement with the literature in measures of Trouton ratio, filament mid-plane radius and field evolution in stress and deformation-rates. Plausibility to discuss such provocative flow features as observed here, has only proved possible by first establishing the present high-precision numerical schemes, which are eminently suitable for the task at hand in solving this extension-dominated free-surface dynamic problem.

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3-modes			
	Mode 1	Mode 2	Mode 3
λ_i (s ⁻¹)	0.421	0.0563	0.00306
μ_i (Pa.s)	25.8	7.71	1.37
α_i (Giesekus)	0.3162	0.2422	0.0993
e (PTT)	0.035	0.035	0.035
ξ	0.13	0.13	0.13
We_i	1.886	0.252	0.014
μ_s (solvent viscosity) (Pa.s)	0.069		
2-modes			
We_i	1.886	0.252	
μ_s (solvent viscosity) (Pa.s)	1.439		
single-mode			
We_i	1.886		
μ_s (solvent viscosity) (Pa.s)	9.149		

Table 1: Material properties for single and multi-mode models.

Re (Reynolds number)	$4.7 \cdot 10^{-4}$
Λ_0 (initial aspect ratio)	0.54
L_0 (initial radius) (m)	$1.89 \cdot 10^{-3}$
χ (surface-tension coefficient) (Nm ⁻¹)	$30.0 \cdot 10^{-3}$
ρ (density) (Kg.m ⁻³)	1030
μ_0 (zero shear viscosity) (Pa.s)	34.949
$\dot{E}$ (extension rate) (s ⁻¹)	4.48
Ca (Capillary number)	9.86
Bo (Bond number)	1.2

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