Sub-cell approximations for viscoelastic flows-filament stretching

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Abstract

The accuracy, stability and consistency of new stress interpolation schemes is investigated, based upon sub-cell approximations. This includes the contrast of two alternative hybrid spatial discretisations: a cell-vertex finite element/volume (*fe/fv*) scheme and a finite element equivalent (*fe*). Here, the interest is to explore the consequences of utilizing conventional methodology and to demonstrate resulting drawbacks in the presence of complex stress equation source terms. Alternative strategies worthy of consideration are presented for a constant shear viscosity model, that of Oldroyd-B, with strain-hardening and unbounded extensional properties. We demonstrate how high-order accuracy may be achieved by respecting consistency in our algorithmic constructions.

Both *fe*- and *fv*-spatial discretisations are embedded within this methodology. Linear interpolation for stress, of either *fe*- or *fv*-form on triangular sub-cells, is referenced within parent triangular finite elements in two dimensions. Finite element discretization is employed for the momentum and continuity system, via a second-order pressure-correction scheme. In this regard, a complex filament-stretching flow with a free-surface is selected to compute the transient evolution of kinematic and stress fields. Shortcomings of various up-winding schemes are discussed, whilst dealing with such free-surface type problems and appropriate strategies for dealing with these types of problems are outlined.

1. Introduction

The literature on *fv*-implementations for viscoelastic flow are extensively reviewed in ref. [14,22]. In previous studies [1,22,23], we have established stable and accurate steady-state finite volume schemes to simulate complex steady viscoelastic flows. A cell-vertex FV-approach is adopted, inspired by the works of Morton and co-workers [6,12], Struijs et al. [19], and Tomaich and Roe [21]. The method is applied on triangular *fe* meshes with *fv* sub-cells, reminiscent of the sub-element *fe* implementation of Marchal and Crochet [9], although their work was implemented on rectangular meshes. Generally, cell-vertex formulations require considerably less degrees of freedom than cell-centred

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forms, and lead to fluctuation distribution stencils to associate cell contributions with nodal equations. Compared to cell-centred methods, cell-vertex schemes maintain their accuracy for broader families of non-uniform and distorted meshes, and are less susceptible to spurious modes than their cell-centred counterparts [12].

Features of the current hybrid approach include a time-stepping procedure that combines a finite element discretisation (semi-implicit second-order/pressure correction) for continuity and momentum equations, with a cell-vertex finite volume scheme for stress. This effective combination is employed as a fractional-staged formulation within each time-step. An individual parent finite element triangular cell is divided into four finite-volume triangular sub-cells. Interpolation on the parent-*fe*-cell is quadratic in velocity, linear in pressure. On the sub-cell, interpolation is linear only. An important aspect of the discretisation is that with stress variables located at the vertices of the finite volume cells, no interpolation is required to recover the finite element nodal stress values.

Under *fv*-discretisation and to date, various forms of fluctuation splitting schemes have been considered [22], and their properties analysed. Our prior studies have established accuracy properties against analytical solution for smooth flows [22], and some non-smooth flows [2]. The relative strengths and weaknesses of various strategies for dealing with flux and source terms have been identified. In our most up-to-date strategy for stress nodal update, a fluctuation distribution contribution over the *fv*-triangle, and a uniform distribution over the median dual cell is proposed. By blending fluctuation distribution and median-dual cell contributions, various scheme combinations may be derived ($\{CT_i\}_{i=0,\dots,3}$ see [2] and [24]). In the current work, the problem chosen is one of filament stretching, a complex transient viscoelastic flow with a compressing free-surface. It is a pure extensional flow, which reflects some of the numerical difficulties involved in solving more generalised viscoelastic flows, where both source and flux terms may contribute equally. The choice of constitutive model is taken as that of Oldroyd-B-form, representing a constant shear viscosity fluid with strain-hardening properties. Here, we have undertaken the study in two distinct phases. First, through spatial discretisation changes (right-hand-side stress equation contributions). Secondly, through consideration of time-term approximations (left-hand-side terms). Under the spatial phase, we present fluctuation distribution (FD), with and without median-dual-cell (*mdc*) contributions. An optimal choice of upwinding factors (α_i^T) for this problem is also recommended. The second phase of the study addresses the discrete approximation of the *fv* time-term, with and without area-weighting factors (α_i^T).

In similar fashion to the foregoing, the hybrid *fe*-scheme is also a sub-cell scheme. Here again, stress is discretely represented in linear form, where the basis of reference is a parent-triangular element, upon which velocity and pressure are defined. Others have also recommended linear finite element representations in the literature. For example, a transient decoupled finite element method was introduced by Basombrio et al. [3], employing a Lagrange-Galerkin technique for velocities. There, stress was approximated using a Lesaint-Raviart discontinuous method with linear shape functions, for the benchmark problem of 4:1 contraction flow and an Oldroyd-B fluid. Here likewise, sub-

element linear stress representation is found to perform in a superior manner to parent-element quadratic interpolation, noted through improved stability properties.

To provide some background, filament stretching flows are common in everyday life. Such a setting may be encountered in rheometry, for example, whilst measuring the elongational viscosity of a fluid [17,20]. Resistance of a material to stretching deformation can be characterised by its uniaxial extensional viscosity. The accurate estimation of extensional viscosity is of considerable importance in many polymer-processing operations. Also, many industrial applications, such as coating are susceptible to instabilities that arise from the elasticity of polymeric liquids. The precise measurement of extensional viscosity is a difficult task to perform for highly-mobile fluids, exacerbated by the presence of shearing effects at solid-liquid boundary interfaces.

Over the preceding decade, much interest has been expressed in the area of extensional rheology for polymer solutions [15,16,27]. The filament-stretching rheometer, introduced by Tritahtmadja and Sridhar [20] has proved to be a useful experimental device. Both experimental and numerical studies have largely contributed to the understanding of extensional rheometry. The primary goal of these studies has been to determine the nature of deformation in the central portion of the filament. Finite element simulations for viscous and viscoelastic fluids within filament-stretching flows has been performed and recorded extensively in the literature [8,15,16,25]. A detailed literature review on this problem is presented in our precursor work [5]. Here, we summarise some key references in the context of viscoelastic fluids. McKinley et al. [11] observed in their experiments, on PIB based Boger fluids, that the liquid-bridge partially decohered from one of the end-plates (fixed, bottom-plate) at high strain-rates (so, for $De > 1$). Such instability was observed to begin as a non-axisymmetric deformation near the end-plates beyond a certain critical strain-level. Subsequently, the fluid column would break into fibrils, upon further elongation. The critical strain for the onset of this instability was found to decrease with increasing De number. Also, these authors reported that Newtonian-liquid bridges of similar viscosity, did not display such instabilities. Spiegelberg and McKinley [18] generated numerical predictions for stretched filaments of Oldroyd-B fluids, using commercially-based software (POLYFLOW). These authors observed that the initial flow-inhomogeneity stimulated the formation of a stress boundary layer near the free-surface of the liquid-bridge at large-strains. For Hencky strains of $\epsilon = 4^+$, it was difficult to represent the rapidly draining region near the end-plates. Significantly, onset of elastic instability appeared close to $\epsilon=4$ for large Deborah numbers.

Rasmussen and Hassager [15] conducted a three-dimensional Lagrangian analysis for filament-stretching upon a UCM fluid. Surface-tension, inertia and gravity were ignored in their numerical calculations. Again, these authors observed the formation of a stress boundary layer near the filament free-surface. Due to numerical approximation errors, their analysis was restricted to Hencky-strains of less than 2.5 ($De > 1$). Sizaire and Legat [16] conducted a numerical study of a filament-stretching device, considering both Newtonian and viscoelastic models. A mixed-finite element method with a conventional Galerkin weak formulation was employed, with a predictor-corrector scheme for time

integration. The pressure, velocity and stress were approximated by linear, quadratic and 4x4 sub-linear interpolations respectively. The *fe*-scheme used was that of Marchal and Crochet SU-scheme [9]. In their findings, no difference were noted between linear and quadratic interpolations for stress. There, the extensional behaviour of a constant viscosity Boger fluid was emulated with a FENE-CR model, which is less severe to solve than the Oldroyd B model. Both inertia and gravity were ignored. The aspect ratio of the filament was 0.54, and a fluid with moderate to high extensional viscosity was considered. Their results corroborated well with those extracted from the rheometer of Tirtaatmaja and Sridhar [20]. The finite element scheme reported in the present work has linear sub-cell form, similar to algorithms of [9,16].

In the study of complex problems under filament-stretching flows, we proceed to demonstrate improved levels of accuracy achieved with a Lax-construction (*fv* or *fe*), that is second-order accurate in time. Here, we have been able to isolate the shortcomings of typical upwinding schemes, whilst dealing with these transient free-surface problems. The success of our implementations may be judged against the attainment of minimal error in stress and for validation against data reported in the literature.

2. Governing Equations

The governing equations for incompressible, viscoelastic flows are represented by conservation laws for mass and momentum, in conjunction with an equation of state for stress. Posed in non-dimensional form, the balance equations under isothermal flow conditions may be expressed as,

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$Re \frac{\partial \mathbf{u}}{\partial t} = -Re \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \nabla \cdot \left(2 \frac{\mu_2}{\mu_0} \mathbf{d} + \boldsymbol{\tau} \right), \quad (2)$$

and the constitutive equation for an Oldroyd-B fluid [13] may be represented in the form:

$$We \frac{\partial \boldsymbol{\tau}}{\partial t} = -We \mathbf{u} \cdot \nabla \boldsymbol{\tau} - \boldsymbol{\tau} + 2 \frac{\mu_1}{\mu_0} \mathbf{d} + We (\mathbf{L} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}^T). \quad (3)$$

Here, $\mathbf{u}$, p , $\boldsymbol{\tau}$ represent the fluid velocity, the hydrodynamic pressure and the extra-stress tensor, respectively. The total viscosity μ_0 is split into Newtonian (μ_2) and polymeric (μ_1) contributions, so that $\mu_0 = \mu_1 + \mu_2$; $\mathbf{d} = (\mathbf{L} + \mathbf{L}^T)/2$ represents the Eulerian rate-of-deformation tensor and $\mathbf{L}^T = \nabla \mathbf{u}$, the velocity gradient. Non-dimensional variables, namely, velocity, pressure, length, stress and time may be interpreted via appropriate scales,

$$\mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad p^* = \frac{L}{\mu_0 U} p, \quad x^* = \frac{x}{L}, \quad \boldsymbol{\tau}^* = \frac{L}{\mu_0 U} \boldsymbol{\tau}, \quad \mu^* = \frac{\mu}{\mu_0}, \quad t^* = t \frac{U}{L}. \quad (4)$$

Group numbers are Reynolds, Weissenberg and Capillary numbers, defined viz.,

$$\text{Re} = \frac{\rho UL}{\mu_0}, \text{We} = \frac{\lambda U}{L}, \text{Ca} = \mu_0 \frac{U}{\chi} \quad (5)$$

where, ρ , λ and χ represent fluid properties of density, relaxation time, and surface tension coefficient. Then, U ($U = \dot{\epsilon}_0 L$), L are characteristic velocity and length scales of the problem. This assumes an initial stretch-rate of $\dot{\epsilon}_0$.

3. Numerical schemes

To solve the related governing equations, two time-dependent finite element and hybrid finite element/volume discretisations are considered. Both schemes are based on a high-order semi-implicit pressure-correction algorithm (TGPC) [10] that employs a two-step Lax-Wendroff approach, and a Taylor series expansion up to second-order in time. A three fractional-staged equations system emerges. A non-solenoidal velocity field is computed at stage one via a predictor-corrector scheme. At stage two, temporal pressure-difference on the time-step is computed, via a Poisson equation. Finally, a solenoidal velocity field is recovered at stage three to complete the time-step loop, see Refs. [10,22] for details.

The resulting space-time discrete matrix-vector system extracted from the general form of the scheme may be expressed as,

$$\begin{aligned} \text{Stage 1a:} \quad & A_u(U^{n+\frac{1}{2}} - U^n) = b_u(p^n, U^n, T^n, D^n), \\ & \frac{2}{\Delta t} A_\tau(T^{n+\frac{1}{2}} - T^n) = b_\tau(U^n, T^n, D^n), \\ \text{Stage 1b:} \quad & A_u(U^* - U^n) = b_u(p^n, U^n, U^{n+\frac{1}{2}}, T^{n+\frac{1}{2}}, D^{n+\frac{1}{2}}), \\ & \frac{1}{\Delta t} A_\tau(T^{n+1} - T^n) = b_\tau(U^{n+\frac{1}{2}}, T^{n+\frac{1}{2}}, D^{n+\frac{1}{2}}), \\ \text{Stage 2:} \quad & A_2(p^{n+1} - p^n) = b_2(U^*), \\ \text{Stage 3:} \quad & A_3(U^{n+1} - U^*) = b_3(p^n, p^{n+1}), \end{aligned} \quad (6)$$

where

$$\begin{aligned} A_u &= \frac{2\text{Re}}{\Delta t} M + \frac{\mu_2}{2} S, \quad b_u = \{F_1 - [\mu_2 S + \text{Re} N(U)]U - B_1 T_1\}^n + L^T P^n, \\ A_2 &= K, \quad b_2 = \frac{2\text{Re}}{\Delta t} L U^*, \quad A_3 = \frac{2\text{Re}}{\Delta t} M, \quad \text{and} \quad b_3 = -L^T (P^{n+1} - P^n). \end{aligned}$$

Above, M , S , and $N(U)$ represent the consistent mass-matrix, the diffusion matrix and the advection matrix, respectively. Throughout the problem domain Ω , these matrices are expressed as,

$$M_{ij} = \int_{\Omega} \phi_i \phi_j d\Omega, S_{ij} = \int_{\Omega} \nabla \phi_i \nabla \phi_j d\Omega, N(U)_{ij} = \int_{\Omega} \phi_i (\phi_l U_l) \nabla \phi_j d\Omega. \quad (7)$$

The pressure stiffness matrix, K , and remaining matrices adopt forms:

$$K_{ij} = \int_{\Omega} \nabla \psi_i \nabla \psi_j d\Omega, (L_k)_{ij} = \int_{\Omega} \psi_i \frac{\partial \phi_j}{\partial x_k} d\Omega, (B_1)_{ij} = \int_{\Omega} \nabla \phi_i \phi_j d\Omega. \quad (8)$$

Stress-related matrices and vectors (A_{τ} and b_{τ}) are described for each scheme below. In the momentum equation, diffusion terms are approximated with a Crank-Nicolson discretisation over a time-step, introducing implicitness and stability to the scheme. Velocity on the parent fe -cell is represented through piecewise-continuous quadratic interpolation ($\phi_j(\mathbf{x})$), whilst that on pressure is linear ($\psi_j(\mathbf{x})$). Such interpolation may be represented in nodal summary, viz.

$$\mathbf{u}(\mathbf{x}, t) = \sum_j \phi_j \mathbf{U}_j, p(\mathbf{x}, t) = \sum_j \psi_j P_j. \quad (9)$$

An accelerated Jacobi iteration is employed to solve the resulting algebraic equations at stages one and three, where no explicit assembly of system matrices is required. It demands only a few iterations. This efficient element-by-element procedure is space optimal, reflecting overall linear space complexity in nodal degrees of freedom, and linear time-complexity per time-step. The Jacobi scheme is an appropriate choice, being well-suited for parallelisation (see Grant et al. [7]) and for these mass-matrix equations, along with their suitable conditioning properties. In contrast, a direct scheme Choleski reduction is employed at stage two, to compute the pressure-difference over the time-step. This choice upholds the order of efficiency of the overall scheme and avoids iterative solution for this stage, where conditioning may be a serious issue to contend with.

Sub-cell finite element stress discretisation:

In the pure finite element implementation, the stress equation matrix-vector components are defined as,

$$A_{\tau} = M_{sc}, \\ b_{\tau} = \frac{2\mu_1}{We} M_{sc} \mathbf{D} - \left(\frac{1}{We} M_{sc} + N_{sc}^{(1)}(\mathbf{U}) \right) \mathbf{T} + (N_{sc}^{(2)}(\mathbf{T})(\nabla \mathbf{U}) + (N_{sc}^{(2)}(\mathbf{T})(\nabla \mathbf{U}))^{\dagger}) \quad (10)$$

where for a sub-cell element Ω_{τ} of Fig. 1, the relevant matrices are given by

$$(M_{sc})_{ij} = \int_{\Omega_T} \omega_i \psi_j d\Omega, \quad (N_{sc}^{(1)}(U))_{ij} = \int_{\Omega_T} \omega_i (\psi_l U_l) \nabla \psi_j d\Omega,$$

$$N_{sc}^{(2)}(T)_{ij} = \int_{\Omega_T} \omega_i (\psi_l T_l) \psi_j d\Omega.$$

In the above, the weighting function, ω_i , may assume alternative forms:

Galerkin: $\omega_i = \psi_i$

SUPG: $\omega_i = \psi_i + \alpha^h \mathbf{u} \cdot \nabla \psi_i$, where α^h is defined elsewhere (see Carew et al [4])

$$\text{Lax-form: } \omega_i = \psi_i + \frac{\Delta t}{4\Omega_T} \mathbf{a} \cdot \mathbf{n}_l^T \quad (\text{see } fv\text{-schemes below}). \quad (11)$$

The implementation of these alternatives is discussed below.

Sub-cell finite volume stress discretisation:

Cell-vertex *fv*-schemes applied to stress are based upon an upwinding technique (fluctuation distribution), that distributes control volume residuals to provide nodal solution updates. Briefly, by rewriting the stress constitutive equation (3) with convective flux ($\mathbf{R}$) and source ($\mathbf{Q}$), one may obtain:

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = -\mathbf{R} + \mathbf{Q}, \quad (12)$$

$$\mathbf{R} = \mathbf{u} \cdot \nabla \boldsymbol{\tau}, \quad (13)$$

$$\mathbf{Q} = \frac{1}{We} \left(2 \frac{\mu_l}{\mu} \mathbf{d} - \boldsymbol{\tau} \right) + \mathbf{L} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}^T. \quad (14)$$

We consider each scalar stress component, τ , acting on an arbitrary volume Ω , whose variation is controlled through scalar components of flux and source,

$$\frac{\partial}{\partial t} \int_{\Omega_l} \tau d\Omega = - \int_{\Omega_l} R d\Omega + \int_{\Omega_l} Q d\Omega. \quad (15)$$

The goal is to evaluate these flux and source variations on each finite volume triangle, and to distribute the same to the three vertices of the cell, according to the preferred strategy. The update of a given node l is obtained by summing the contributions from its control volume Ω_l , which is composed of all the *fv*-triangles surrounding node l , see Fig. 1b. The flux and source residuals may be evaluated over different control volumes. Namely, over the *fv*-triangle $T(\mathbf{R}_T, \mathbf{Q}_T)$ and/or the median dual cell (*mdc*), associated with a given node l within the *fv*-cell T ($\mathbf{R}_{mdc}$, $\mathbf{Q}_{mdc}$). Median-dual-cell zones are non-overlapping regions, defined per node l , an area one-third of the triangular cell over which it is constructed, see Fig. 1.

The consistent distribution approach devised by Webster et al. [24] may be expressed in generalised form, as:

$$\frac{\Delta \tau_l^{n+1}}{\Delta t} = \underbrace{\frac{\sum_{\forall T_l} \delta_T \alpha_l^T b_{T_l}}{\Omega_1}}_{FD} + \underbrace{\frac{\sum_{\forall mdc_l} \delta_{mdc}^T \alpha_{mdc}^T b_{mdc}^l}{\Omega_2}}_{mdc}, \quad (16)$$

where

$$b_{T_l} = (R_T + Q_T), \quad b_{mdc}^l = (R_{mdc} + Q_{mdc})^l.$$

We develop alternative choices for Ω_1 and Ω_2 below, by appealing to definitions,

$$\Omega_{FD} = \sum_{T_l} \delta_T \alpha_l^T \Omega_{T_l}, \text{ and } \Omega_{mdc} = \sum_{mdc_l} \delta_{mdc}^T \Omega_{mdc_l}^T.$$

The parameters, δ_T and δ_{mdc}^T can be combined to derive alternative implementations of various properties. For example, $\delta_T = 1$ and $\delta_{mdc} = 0$ reverts to a pure fluctuation distribution form. Theoretically, δ_T and δ_{mdc} , are mutually linked and complementarily exclusive. Blending FD and *mdc* contributions with parameters δ_T and δ_{mdc} , provides various alternative scheme combinations (CT₀, CT₂, CT₃, see [2,24]). For example, we obtain the CT₃-scheme by demanding $\Omega_1 = \Omega_2 \equiv \Omega_{FD} + \Omega_{mdc}$; alternatively, we can recover the CT₂-scheme with the combination $\{\Omega_1 = \Omega_{FD} \text{ and } \Omega_2 = \Omega_{mdc}\}$; the original nodal-update form (CT₀) corresponds to $\Omega_1 = \Omega_2 \equiv \hat{\Omega}_l$. The consistent approach CT₃ is found to provide high-order accuracy for steady-state flows, and a consistent *mdc* contribution aids stability in time-stepping convergence for complex flows with a significant shear component, such as arise in contraction flows.

Fluctuation distribution coefficients α_l^T , may be provided via an appropriate choice of scheme: for example, LDB, Lax-Wendroff, or PSI. Here, we focus on the two former versions. These are compact-stencil upwinding schemes, that update the solution in a triangular cell. This is accomplished by splitting flux contributions to respective vertices, according to the prescribed strategy, and based upon the properties of conservation, positivity and linearity preservation.

Lax-Wendroff (Lax) scheme: This option is second-order and linear, satisfying linearity-preservation. It is spatially-centered and second-order accurate in space. In addition, it contains a dissipation term, designed to control oscillations in the neighbourhood of discontinuities, thus conferring second-order accuracy in time. For the Lax-scheme, the distribution coefficients α_l^T may be expressed as:

$$\alpha_l^T = \frac{1}{3} + \frac{\Delta t}{4\Omega_T} \mathbf{a} \cdot \mathbf{n}_l^T, \quad (17)$$

where, Δt is a time-step size, $\mathbf{a}$ an averaged cell-advection velocity, and $\mathbf{n}_l^T$ a scaled inward-pointing normal vector to an edge of triangle T, opposing node l .

LDB scheme: This is an alternative to Lax above. It is a linearity-preserving (non-positive) scheme, defined on each fv-triangle by the angles $\gamma_i^\dagger$, subtended at an inflow vertex by the advection velocity $\mathbf{a}$ (an average vector per cell). The LDB distribution coefficients $\alpha_i^\top$, defined per cell node i , may be expressed as:

$$\begin{aligned}\alpha_i &= (\sin \gamma_1 \cos \gamma_2) / \sin(\gamma_1 + \gamma_2), \\ \alpha_j &= (\sin \gamma_2 \cos \gamma_1) / \sin(\gamma_1 + \gamma_2), \\ \alpha_k &= 0.\end{aligned}\tag{18}$$

We point out that the closer the advection velocity $\mathbf{a}$ is to being parallel to a cell boundary, the larger the contribution to the downstream node at that boundary.

4. Problem specification

In this article, we particularly consider the extensional deformation of a viscoelastic filament between two coaxial discs, drawn apart in time at an exponential rate. The context is one where it is reasonable to ignore the effects of inertia and gravity. The plates are retracted, at a constant stretch-rate $\dot{\epsilon}_0$, satisfying the following relationships, on filament length and plate-velocity:

$$\text{Length:} \quad L(t) = \pm(L_0/2) \exp(\dot{\epsilon}_0 t), \tag{19}$$

$$\text{Velocity:} \quad U_z(t) = \pm(\dot{\epsilon}_0 L_0/2) \exp(\dot{\epsilon}_0 t), U_r(t) = 0. \tag{20}$$

Here, $\dot{\epsilon}_0$ is an imposed initial stretch-rate and L_0 is the initial length of the filament. The aspect ratio of the liquid-bridge is defined as $\Lambda_0 = L_0/R_0$, where R_0 is the initial filament radius. We have chosen to report on two filament aspect-ratios, $\Lambda_0 = 2/3$ and $1/3$, following reference [25]. Two types of fluids are employed, one with a high polymeric to solvent viscosity ratio (373.9); and a second, with a relatively low ratio (0.0926). The set of operating parameters are described in Table 1, taking representative data from [26,29] for polyisobutylene-based fluids. As above, in this study we employ the Oldroyd-B model throughout. The boundary conditions and geometric domain considered are illustrated in Fig. 2a. Dirichlet-type boundary conditions are imposed on all but the free-surface. In addition, initial conditions are taken as quiescent, with the exception of those on the moving-plates, where the initial impulsive velocity is taken as $(V_0 = L_0 \dot{\epsilon}_0/2)$.

In order to find the eventual position of the free-surface, we adopt the dynamic condition on the free-surface position function $h(z, t)$, of:

$$\frac{\partial h}{\partial t} = u_r - u_z \frac{\partial h}{\partial z}, \quad \forall t. \tag{21}$$

[†] The equivalent area-weighted version is termed the LDA-scheme

To retain, smoothness and consistency in surface profiles, a Crank-Nicolson treatment (with $\theta=0.5$) is introduced into the discretisation of the second-term of the r.h.s. of Eq. 21, implemented in time-split form as,

$$\frac{\partial h^{t^{n+1}}}{\partial t} = u_r^{t^n} - u_z^{t^n} \left[\left(\frac{\partial h}{\partial z} \right)^{t^{n-1}} + \theta \left\{ \left(\frac{\partial h}{\partial z} \right)^{t^n} - \left(\frac{\partial h}{\partial z} \right)^{t^{n-1}} \right\} \right]. \quad (22)$$

Surface tension, acting upon the free-surface, may be included through the following condition, expressed in non-dimensional form,

$$(-p\mathbf{I} + \boldsymbol{\tau}) \cdot \mathbf{n} = (H / Ca - p_a) \mathbf{n}.$$

Here, $\mathbf{n}$ is the unit normal to the surface, H is its mean Gaussian† curvature, and p_a is the ambient pressure. We observe in Table 1, that Ca^{-1} values are small, being 0.02 and 0.007 for low-polymeric and high-polymeric viscosity cases, respectively. This leads to only minor influence of the surface tension on resultant kinematics, particularly at larger levels of Weissenberg number (see Yao et al. [29]).

A typical rectangular initial mesh of 20 x 100 elements, with 8025 nodes and 4000 triangular elements, is used to perform the numerical computations. The mesh must adjust in time, being stretched and distorted, its movement being controlled by the time-step and remeshing algorithm discussed below.

The normal force (F_z) exerted at the moving end-plates is calculated through the integral of stress over the end-plate area A , viz.,

$$F_z(t) = \int_A (T_{zz} + p) dA, \quad (23)$$

where $T_{zz} = 2\mu_2 D_{zz} + \tau_{zz}$ represents the total stress.

Algorithmic steps

For the transient solution of the stress constitutive equation, we have enjoyed most success with these sub-cell schemes, hybrid *fe/fv* and *fe*, utilising Lax-stencils. From some initial flow field and domain, the free-surface and moving boundary is advanced, according to the flow state upon such boundaries. At each step, the mesh is adjusted to comply with kinematic conditions of Eq (22), that match the motion of the moving free boundary to the velocity field. The kinematic ($\mathbf{u}, p$) iteration (Step 5 below), which is a natural, separate solution stage for transient free-surface problems, is invoked at each movement of the plate. This ensures accuracy in the field solution, alongside local filament-disc movement to accommodate for non-axial domain shifts.

Remeshing algorithm

We have adopted a fixed connectivity approach here, which maintains the original sparsity pattern, associated with the matrices representing the discrete equations.

Distribution of interior nodes may be achieved by using logarithmic or tan-hyperbolic coordinate transformations. Due to the underpinning boundary conditions and the presence of the free-surface, the mesh experiences large deformation both radially and axially. Uniform remeshing is performed radially. For axial remeshing, where the mesh undergoes large stretching, a modified logarithmic remeshing rule is introduced, so that the nodes remain concentrated in certain regions of reference. This strategy has the ability to maintain element density and mesh-aspect ratio without adding or removing elements. Since the shape of the free-surface is highly curved near the moving-plates, a finer mesh is found necessary there, to retain mesh aspect-ratio and hence, accuracy and stability. In this context, we recall the remeshing algorithm implemented in our previous work (see Chandio et al. [5]). In this manner and by design, high element density is maintained near the moving-plates.

In order to conserve the total volume of the filament, we adopt a novel strategy, that we term ‘local disc movement’. This is performed once at each Hencky-strain step. After the movement of the plates, in addition to the movement of nodes in the axial direction, we also allow for movement in the radial direction (r). With translation in both directions over the time-step, as illustrated in Fig. 2c, we ensure each local filament-disc conserves its volume. Hence, the total fluid volume, being the sum of all volume slices, is conserved utilising this localised stratagem.

For any single Hencky-strain step (Δt_{Hencky}), we appeal to the following sequence of algorithmic steps:

- Step 1. Update Hencky-strain, $t_{n+1} = t_n + \Delta t_{Hencky}$
- Step 2. Fix plate-boundary conditions at t_{n+1} (see Eq. 20)
- Step 3. Shift plate locations through a single step (Δt_{Hencky}).
- Step 4. Update mesh-points locations, to occupy new positions relative to plate-movement (axially and radially).
- Step 5. Compute current velocity and pressure field on the shifted domain, solving fractional-stages, with dynamic boundary conditions (see Eq. 21).
- Step 6. Update velocity, pressure, and stress fields alongside free-surface computation, solving field fractional-stages, with dynamic boundary conditions synchronously, through a set number of steps, ($M = \Delta t_{Hencky} / \Delta t_{inner}$) time-steps.
- Step 7. Determine local time of stretch; if not at terminating time, goto Step 1.

At Step 5, velocity and pressure field solutions are determined over each Hencky-strain step to a specified iterative tolerance (normally 10^{-5} to 10^{-6}). We have found no difficulty in decreasing the tolerance at this algorithmic step to extract a fixed-point solution, as the Reynolds number is small (inertia-less). The pseudo-sub-steps demanded at this stage are termed inner-steps, usually $O(10^2)$, but may take up to (10^3) in number (N). In Step 6, we evaluate primary variables ($\mathbf{u}$, p , $\boldsymbol{\tau}$, h) in a synchronous manner, so that, the total number of steps employed (M) corresponds to the ratio of the Hencky-strain step Δt_{Hencky} to the local inner time-step Δt_{inner} .

Dynamic time-step adjustment

In addition, we have implemented a dynamic, problem specific, time-step adjustment procedure, satisfying the CFL condition. For a mesh employed with an initial time step, say $\Delta t_0 = 10^{-3}$, the time-step at time t^{n+1} is chosen so that the distance of the end-plate movement between two consecutive Hencky-strain steps remains constant between two consecutive time-steps. This relation may be expressed as

$$\Delta t_{n+1} = \Delta t_n e^{-\dot{\varepsilon}_0 \Delta t_n}. \quad (23)$$

This procedure has the additional benefits of tailoring the exponential spatial adjustment, natural to the problem, onto the time-shifts employed. Hence, indirectly influencing mesh adjustment.

5. Results and Discussion

Accurate velocity gradient determination is an important aspect to this problem. In this respect, we invoke a combination of field recovery and pointwise boundary procedures. Our earlier devised procedures on recovered velocity-gradients [10], are found to work satisfactorily at all boundaries, but may be improved upon at the end-plates, in terms of pointwise estimation of shear components, D_{rz} (normal components vanish analytically). This retains the unique identity of the particular surface where approximation is applied (discontinuous from surface-to-surface). We are careful not to impose strong (pointwise) continuity conditions at the radial filament centre. Also, the mesh structure in the domain corners is adjusted in such a manner that corner nodes gather contributions from at least two elements (hence, avoiding locking-corners). We note that the quantities plotted are in non-dimensional form only.

Under current settings, it is our experience that surface tension has little effect. This may be due in part to the range of Hencky-strains under consideration (up to unity) and the relatively high level of Weissenberg number (5.08) involved. These observations are in keeping with those reported in Yao et al. [29]. Hence, for present purposes, we choose to neglect surface tension effects.

5.1 Finite volume results

Fluctuation distribution (Lax or LDB)

To check the correctness of our schemes, we focus attention almost entirely upon the distribution of extra-stress (τ_{zz}) along the centreline ($z = 0$). Here, extra stress is the primary variable computed within the *fv*-scheme. At a Hencky-strain of $\varepsilon = 0.2$, the magnitude of stress is a constant near the centreline and drops dramatically near the free-surface node due to the formation of a boundary layer [28]. We may compare two cell-vertex fluctuation distribution schemes of interest, LDB (linear and linearity preserving, second-order for steady problems) and *fv*-Lax (linear, central-scheme with dissipation term, second order). Such FD-schemes are applied consistently to both flux and source terms, by choosing $\delta_{\text{mdc}} = 0$ and $\alpha_l = 1$ on the left hand side of Eq. (16). In Fig. 3a, the resulting trend illustrated via the *fv*-Lax-scheme agrees well with that reported by Yao et al. [28]. With the LDB-scheme, the value of extra-stress at node n_1 vanishes. See Fig. 2b

for the position of node n_1 . The reason for this relatively poor performance of the LDB-scheme can be attributed to the prevailing direction of the radial velocity, V_r , which is directed inwards at node n_1 at all times. With the use of a pure upwinding scheme (such as LDB), by definition of the scheme itself, the nodes on the free-surface boundaries will receive no extra-stress contribution whatsoever. This is so particularly for node n_1 which is the mid-node. In the plot describing the transient stress development (Fig. 3b), we observe that with the LDB-scheme, the value of extra-stress at node n_1 , does not shift away from zero, as indeed it should.

In Fig. 4, we plot the total stress T_{zz} along the centreline of the liquid bridge ($r=0$) versus axial distance z , at a Hencky-strain level of $\varepsilon = 0.2$. Here, we employ the Lax-scheme version, and contrast our results with those of Yao and McKinley [25]. The total stress T_{zz} vanishes at the plates, reaching its maxima at the centre of the filament. We observe close agreement between the two results (to within $O(5\%)$), which visually overlap each other.

Effect of using mdc on rhs

In some of our earlier studies with the hybrid fe/fv algorithm, adding mdc contributions to the fluctuation distribution components, imbued the scheme with stability and accuracy for both steady and transient problems, where shear components were present. In this section, we demonstrate the effect of adding the mdc contributions in the context of this temporal extension-dominated free-surface filament-stretching problem. The scheme employed here is CT_3 , the more consistent variant, with $\delta_{mdc} = 1 - \delta_t$. From Fig. 5, it is immediately apparent that any inclusion of mdc -terms degrades the solution. The magnitude of stress τ_{zz} at $\varepsilon = 1.0$, is minute compared to that for the alternative pure fluctuation distribution implementations. Hence, the preference to discard dissipative mdc -contributions.

5.2 Finite element results and across schemes (fe and fv)

We proceed, in contrast to the above, next to consider the sub-cell finite element schemes: Galerkin, SUPG and fe -Lax. In Fig. 6, we show the development of τ_{zz} stress up to a Hencky-strain level of 0.2 for the mid-plane, free-surface mid-point node at $z = 0$ (node n_1). Both the Galerkin and the fe -Lax schemes provide an increasing stress τ_{zz} trend with increasing Hencky-strain. In contrast, when employing the SUPG scheme, τ_{zz} decreases with time sustaining negative solution values. Here, as with LDB- fv -scheme, no contribution (via the weighting function) is made to the nodes near the free-surface regions, due to the inward pointing direction of the radial velocity. The resulting weighting function becomes negative, whilst that for the Galerkin or fe -Lax scheme remains positive. The stress τ_{zz} should increase at this point and the surrounding region (see, for example, Yao and McKinley [25] and Yao et al. [28]). Hence, the SUPG-form is not considered in deliberations below.

Next, we compare and contrast solutions generated across schemes, with our different sub-cell implementations. We commence with the finite element based schemes, and then switch attention to finite-volume alternatives. The aim is to extract the benefits of each scheme, and thereupon, develop an overall improved form. Suitable candidates present themselves under finite element form of: Galerkin, and *fe*-Lax; and under finite volume form of: pure Lax fluctuation distribution, with and without area-weighting on the *lhs*. Starting with the low Hencky-strain level of $\varepsilon = 0.2$, we demonstrate that comparable results for *fe* and *fv* implementations are obtained. This is exposed in Fig. 7a and Fig. 7b for low and high-polymeric viscosity fluids, respectively, indicating consistency across these scheme choices.

Accuracy in time

Concentrating on *fe*-Lax, in Fig. 8a, we show that at the low Hencky-strain level of $\varepsilon = 0.2$, convergence is achieved using a fixed high Hencky-strain time-step movement of $\Delta t_{Hencky} = 10^{-3}$. Beyond the time-step level of 10^{-3} , there is barely any difference between solutions corresponding to different time-steps of 10^{-3} and 10^{-4} . Along the axial mid-plane up to radial distance $r = 0.7$, the extensional stress component τ_{zz} remains constant at 0.013 units. It decreases radially thereafter towards the free-surface mid-point to a value of 0.005 units. In contrast, moving on to the relatively high Hencky-strain level of $\varepsilon = 1.0$, it is necessary to reduce Δt_{Hencky} to 10^{-4} to achieve an acceptable level of accuracy. As displayed in Fig. 8b, there is little difference near the axial mid-plane, and the stress τ_{zz} level lies around 0.24 units for all Hencky-strain time-steps employed. In contrast, near the free-surface mid-point, stress values attained equate to 0.005, 0.21 and 0.25 units for $\Delta t_{Hencky} = 10^{-2}$, 10^{-3} and 10^{-4} , respectively. Greater temporal accuracy is necessary for the stress to sufficiently develop around the free-surface region, over that required at the radial centreline of the filament. Δt_{Hencky} of 10^{-4} is sufficient for this purpose, as clarified by even further reduction.

To allow for more computation on the free-surface, the *synchronous* procedure with sub-iteration has also been found useful. In Fig. 9 at $\varepsilon = 1.0$, we amplify the influence of the inner local time-step $M = \Delta t_{Hencky} / \Delta t_{inner}$, by adopting the lower Hencky-strain step $\Delta t_{Hencky} = 10^{-2}$, which exposes the differences involved. We also include data here, for the more stringent $\Delta t_{Hencky} = 10^{-4}$, which clarifies the lessening influence of sub-iteration as Δt_{Hencky} is reduced. We may observe that the target (more accurate) solution for comparison is provided more sharply by the α -area-weighting scheme below. There, one gathers that it is important to first exert control through Δt_{Hencky} , only after which one may appeal to sub-iteration. We increase sub-iteration number M from 10 through 50, 100, 500 and 1000, for stress equation calculations, displaying τ_{zz} along the filament axial mid-plane. This covers data for *fe*-Lax in Fig. 9a and *fv*-Lax in Fig. 9b. At $\Delta t_{Hencky} = 10^{-2}$, we observe that stress solution changes significantly in the vicinity of the

free-surface, by increasing the ratio M from 10 through 500. Above $M=500$, little further adjustment is observed. Hence, a maximum of $M=500$ is found sufficient. In contrast, around the radial centreline of the liquid bridge, the stress component is convergent at all sub-iteration settings, $M=10$ to 10^3 . These observations hold true for all our Lax-type implementations: that is, *fe*-Lax, Galerkin-*fe*, and *fv*-Lax. At $\Delta t_{Hencky} = 10^{-4}$, one notes that $M=10$ is sufficient throughout, reinforcing above comments over precedence of parameter settings.

All results are based on mesh M1, biased towards the moving-plates with high element densities. This follows our prior mesh refinement studies within this range of Hencky-strains for Newtonian fluids [5], and provides a high degree of localised solution resolution throughout the stretching procedure. Nevertheless, we have also employed mesh M2, of uniform structure at the start of the stretching, $t=0^+$. We have found comparable results between these two meshing options, with barely any difference in solution variables away from the end-plates.

Effect of area-weighting factor (α_l^T -lhs) on stress equation

We proceed to discuss the importance of the transient (left-hand-side) terms of the stress equation. These may be modified in the *fv*-scheme according to (CT3-area, using appropriate α_l^T parameters). As we have observed above, temporal accuracy may be achieved either by reducing time-step (Hencky-strain step), or by increasing the number of sub-iterations M for velocity, pressure and stress. The difficulty arising in these *fe* and *fv*-schemes, is that stress development near the free-surface region in the centre of the filament often lacks sufficient accuracy. We have found by empirical investigation, that by incorporating the area-weighting factor α_l^T on the left-hand-side (governing transient terms) of the standard flux distribution *fv*-scheme, the stress component τ_{zz} acquires greater contributions near the free-surface. Then, stress development in that vicinity is acceptable, even with larger Hencky-strain steps of $\Delta t = 10^{-2}$. With back-reference to Fig. 5, we contrast solution representation between the standard *fv*-Lax scheme and the modified area-weighting version (α_l^T -lhs incorporated). The extensional stress τ_{zz} is plotted along the axial mid-plane, at a Hencky-strain level of $\varepsilon = 1$. For the standard *fv*-Lax scheme, τ_{zz} is constant near the radial symmetry line and surroundings, at about 0.24 units. Subsequently, it decreases, and continues to do so towards near the free-surface region to a value of 0.065 units. In contrast, with the α_l^T -lhs version, the stress τ_{zz} shows non-monotonic adjustment, first decreasing and then, increasing towards the free-surface to a value of 0.23 units. There is barely any change of behaviour near the radial symmetry line. Here, the stress τ_{zz} value is close to that corresponding to both *fe* or *fv*-scheme results with $\Delta t = 10^{-3}$ and $\Delta t = 10^{-4}$. These differences, observed when the α_l^T area-weighting parameter is incorporated, are present only at larger Hencky-strain levels (longer times) being relatively insignificant at the low level of $\varepsilon = 0.2$ (shorter times). The contribution for stress at the free-surface node is conspicuous in the transient stress development plots, illustrated in Fig. 10. The inclusion of modified area-weighting does not make any contribution to node n_2 (Fig. 10a), which lies on the axial centreline. In

contrast, on the free-surface node (n_1), more stretching is realised with modified area-weighting (see Fig. 10b).

Similarly, to the finite volume implementations with the α_i^T -lhs area-weighting parameter, we have been able to generate comparable results in a modified *fe*-Lax scheme. Below, we consider this modified *fv*- α_i^T -lhs version further, where there is no distinction drawn between the *fv* or *fe* forms used.

Dynamic Hencky-strain step

Here, we contrast two different sets of results, one with a fixed Hencky-strain step (Δt_{Hencky}) and the other with a dynamic time-step (see Eq. 23). The latter Hencky-strain step is chosen, so that the shift of the moving end-plates, between two consecutive Hencky-strain steps, is maintained as a constant. In Fig. 11a, we contrast the τ_{zz} component along the filament mid-plane axis for the two settings. The scheme employed is one of pure fluctuation distribution with a standard setting on the transient-lhs-terms of the stress-equation. The Hencky-strain level in these figures is $\varepsilon = 1$. Similarly, in Fig. 11b, we plot the same two sets when incorporating modified area-weighting on the transient-lhs-terms. From Fig. 11a, it is apparent that the variable time-step gathers greater contribution to the free-surface mid-point, when compared to the fixed time-step scenario. In contrast, with the α_i^T -lhs version, no significant difference is noted in the value of τ_{zz} at the free-surface mid-point node n_1 when incorporating dynamic/adaptive time step adjustment (see Fig. 11b). This finding reiterates the time accuracy issue, we have discussed above, upon using lhs-area-weighting.

5.3 Problem variants

Low and high-polymeric viscosity results ($A_0=2/3$)

In this context, we limit ourselves to one aspect ratio of $2/3$, and we consider two polymeric viscosities (μ_1 / μ_0) at an order of magnitude apart, 0.085 and 0.91 units. At a low Hencky-strain level of $\varepsilon = 0.2$ and both using the same numerical settings, that is $\Delta t_{Hencky} = 10^{-2}$, ($N=300$ for $\mathbf{u}$, p , and $M = \Delta t_{Hencky} / \Delta t_{inner} = 10$ for $\mathbf{u}$, p , $\boldsymbol{\tau}$ and h), accurate solutions have been determined for the low-viscosity case. This is observed through velocity and stress representations, being smooth everywhere on the field (see Fig. 12). In contrast, the more difficult problem scenario with the high viscosity fluid lacks accuracy with these less stringent numerical parameter settings, as anticipated. This may be due to larger stress values generated at the centre of the filament. As displayed in Fig. 13, both components of velocity are oscillatory on the field. To overcome this inadequacy, it is necessary to perform more computation on the kinematic field, say by increasing the pseudo-iteration number N to 1000 (see Step 5 of the algorithm above), or enforcing a tighter tolerance of 10^{-6} . The corresponding Hencky-strain solution is shown in Fig. 14, where both velocity components V_r and V_z are now smooth throughout the domain.

The corresponding high-polymeric viscosity fluid stress component τ_{zz} is plotted in Fig. 15, along the axial mid-plane section as a function of r at a low Hencky-strain level of $\varepsilon = 0.2$. Here, similar behaviour is observed between both viscosity ratio results; a constant value of τ_{zz} is observed throughout a large portion of the radius, followed by a decrease in the free-surface vicinity. The main disparity lies in the values, where for low-viscosity, τ_{zz} is about 0.013 at $r=0$, decreasing to 0.006 at the free-surface. For the high-polymeric viscosity alternative, equivalent values of τ_{zz} are 0.165 and 0.070 units. One notes, the stress level has increased about ten-fold, similar to the ratio in polymeric viscosities from case-to-case (0.91/0.085).

At the higher level of Hencky-strain of $\varepsilon = 1.0$, a Hencky-strain increment $\Delta t_{Hencky} = 10^{-2}$ is sufficient to accurately resolve the low-polymeric viscosity case. In contrast, for the high-polymeric viscosity fluid, it is necessary to reduce Δt_{Hencky} to 10^{-3} to be able to achieve an acceptable level of Hencky-strain of $\varepsilon = 1.0$. For this fluid, employing $\Delta t_{Hencky} = 10^{-2}$ provides satisfactory solutions up to a Hencky-strain of $\varepsilon = 0.7$. Oncemore, stress levels increase in the vicinity of the free-surface to about 8.14 units. The stress is constant (3.16 units) near the radial symmetry axis, as displayed in Fig. 16. In this case, we observe that more stretching is localised around the free-surface region. This is in accordance with that shown by Sizaire and Legat [16]. In their work, the FENE-CR model was employed with moderate to high-polymeric viscosity components (about $\mu_1 / \mu_0 = 0.66$). For the low-polymeric viscosity fluid, τ_{zz} -values are less variable, being 0.24 and 0.23 units, respectively, at the intersection between the mid-plane and the axial centreline axis, (node n_2), and at the free-surface mid-point (node n_1).

Variation in aspect ratio ($\Lambda_0=2/3$, $\Lambda_0=1/3$)

At a low Hencky-strain level of $\varepsilon = 0.2$, results for both aspect ratios are comparable, with little difference apparent between the two cases near the radial symmetry axis, see Fig. 17. As displayed in Fig. 18a, for $\Lambda_0 = 2/3$ case, the stress τ_{zz} is constant along the axial mid-plane for a large part of the radius (value of 0.013), decreasing to a value of 0.006 units at the free-surface mid-point. Similar behaviour is observed for the alternative ratio $\Lambda_0 = 1/3$ problem. Here, the stress slightly increases to a value of 0.14 units, to decrease thereafter, to 0.004 units at the free-surface mid-point (node n_1).

At the larger Hencky-strain level of $\varepsilon = 1.0$, and for the low-viscosity model, the maximum stress τ_{zz} is about 0.24 units for $\Lambda_0 = 2/3$ (see Fig. 18b). The maximum stress level is located near the axial centreline of the filament (node n_2). At $(r = 0, z = 0)$, τ_{zz} is 0.24 units and constant along the axial mid-plane section (between $r = 0$ and $r = R_{min}/3$). It decreases thereafter, to 0.17 units at $r = 5R_{min}/3$, to increase again at the free-surface mid-point (node n_1) to 0.23 units. In contrast, when the initial aspect ratio is decreased to $1/3$, τ_{zz} increases to about 0.87 units. In this case, a maximum is located at the free-surface. The maximum value along the radial symmetry line is about 0.26 units. It starts to increase along the axial mid-plane, from the radial symmetry line to the

free-surface, where it rises to 0.87 units. This is not true for the larger aspect ratio $\Lambda_0 = 2/3$. As displayed in Fig. 19, for the initial aspect ratio of $1/3$, τ_{zz} maxima (about 1.1 units), arise at the free-surface, with slight deviation from symmetry.

The equivalent velocity profile patterns may be described as follows. The radial velocity V_r tends to its minima (absolute maxima) around the vicinity of the free-surface, near the axial mid-plane section, where the filament is thinnest. In contrast, V_r is negligible near the plates and on the radial symmetry line. The axial velocity component V_z , is in comparison, symmetrical around the axial mid-plane section, with minimum absolute value at this location, increasing towards a maximum around the plates (both plates-moving problem), where boundary conditions are imposed. For V_z , and for both aspect ratios, extrema arise of around -1.2 and 1.2 units, respectively. In contrast, V_r minima change dramatically from -0.45 for $\Lambda_0 = 2/3$ to -0.9 for $\Lambda_0 = 1/3$.

To validate our results for the case of $\Lambda_0 = 1/3$, we contrast, in Fig. 20 the velocity component V_r against that of Yao and McKinley [25] along the free-surface line, at $\varepsilon = 1$. As observed by Yao and McKinley, the lubrication approximation well represents the numerical solution of the radial velocity V_r along the free-surface line, at least at low Hencky-strains. Again, we show that our numerical results agree closely with those of Yao and McKinley for the axial velocity component V_z along the centreline of the filament liquid bridge, as indicated in Fig. 21. Here, we observe that the two results are in accordance with the lubrication approximation. For the level of Hencky-strain considered, the Newtonian kinematics do not differ from the viscoelastic counterpart. The normal force exerted on the moving end-plates is plotted in Fig. 22. The normal force assumes a maximum at $t=0^+$, then decreases with the stretching experienced. A similar trend has been reported by Yao and McKinley [25].

The pressure field discloses a minimum value of about -2.0 units, near the intersection between the radial centreline of the liquid bridge and the plates. We also observe localised pressure minima around the centre of the filament, close to the free-surface. This is the case for the initial aspect ratio $\Lambda_0 = 2/3$. For the $\Lambda_0 = 1/3$, the pressure decreases further to -3.2 units, its minima being localised near the free-surface.

In Fig. 23, development in minimum filament radius is plotted against Hencky-strain, up to levels of unity, and compared with the lubrication approximation. Solutions reflect the fact that there is a change in slope due to increased strain-hardening, which becomes more significant for filaments with smaller initial aspect ratios. This point has been recognised and reported upon earlier by others, see Yao and McKinley [25]. So, for example, we have observed at $\varepsilon = 1.0$, $R_{\min}/R_0$ is about 0.47 for the $\Lambda_0 = 2/3$ instance, whilst this quantity decreases to 0.42 for the $\Lambda_0 = 1/3$ case.

6. Conclusion

In this study of a complex viscoelastic (Oldroyd-B model) filament-stretching problem, we have employed both finite element and finite volume schemes for stress,

utilising a linear sub-cell structure. Two material settings have been analysed, low and high polymeric viscosity ratio models, with two initial aspect ratios of one-third and two-thirds. We summarise our main conclusions. We observe that certain upwinding schemes fail to perform well under such a strongly extensional transient flow, due to the restrictions imposed by the velocity field. On the hybrid fv -side, we have displayed improved levels of accuracy with a second-order Lax-Wendroff scheme, in contrast to an LDB-implementation. Also, mdc -inclusion is not recommended for such transient extension-dominated problems with contact-line/interfaces, as it degrades accuracy in the transient. In the case of the counterpart fe -scheme, SUPG performs poorly, in a similar fashion as to the LDB in the fv -implementation. We have been able to demonstrate how accuracy in time and space for stress may be achieved through a modified area weighting scheme (α_i^T -lhs version) for either fe or fv - schemes, at low-cost in time ($\Delta t_{Hencky} = 10^{-2}$). For the standard fluctuation distribution scheme, improved accuracy has been achieved by using dynamic/adaptive time-step adjustment. This ploy may be used effectively in any of our schemes. We have corroborated our results against the literature and found close agreement. The very nature of the filament-stretching problem (dynamic, free-surface, with compression) has lent itself to well-characterise suitable numerical schemes. In particular, discounting those of an inadequate upwinding nature. This is viewed as a viable contribution to the development of numerical schemes more widely.

Innovations introduced into the algorithmic steps are: a local disc movement procedure, in conjunction with a Thompson remeshing strategy, to conserve the total volume of the filament. Synchronisation for $\{\mathbf{u}, p, \boldsymbol{\tau}, h\}$, with the local-disc movement procedure for volume conservation, superceeds alternative strategies. A sub-iteration on the kinematic field is also found to be of benefit. We have been able to achieve a high degree of precision on velocity gradients with pointwise estimation on the plates, and recovery elsewhere. Overall, a combination with all these aspects taken into account, has been important to derive accurate determination of the dynamically evolving stress.

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- Fig. 1: (a) fe with 4 fv -sub-cells Ω_T , (b) mdc area for node l .
- Fig. 2: Domain of filament, full length model with boundary conditions, and local disc movement definition.
- Fig. 3: Comparison between Lax and LDB schemes: a) τ_{zz} along the mid-plane section of the filament at $\varepsilon = 0.2$, b) transient development of τ_{zz} in time for the free-surface mid-point (node n_1), $\varepsilon = 0$ to 0.2 .
- Fig. 4: Profiles of total stress T_{zz} along the radial centreline of the liquid bridge, comparison with Yao and McKinley [25], at a Hencky-strain level of $\varepsilon = 0.2$, low-polymeric viscosity fluid.
- Fig. 5: Comparison of different fv -schemes, fv -Lax, modified area-weighting ($\alpha_l^T - lhs$ version) and CT3, τ_{zz} along the mid-plane section of the filament at a Hencky-strain level of $\varepsilon = 1$, low-polymeric viscosity fluid.
- Fig. 6: Transient development of τ_{zz} for the free-surface mid-point node (n_1), up to a Hencky-strain level of 0.2 , comparison of fe -based schemes (Galerkin, SUPG and fe -Lax).
- Fig. 7: Comparison between fv -Lax and fe -Lax schemes, at a Hencky-strain level of $\varepsilon = 0.2$; τ_{zz} along the mid-plane axis of the filament ($z=0$); a) low-polymeric viscosity fluid, b) high-polymeric viscosity fluid.
- Fig. 8: Profiles of τ_{zz} along the axial mid-plane axis of the filament, fe -Lax, accuracy in time for different Hencky-strain steps (10^{-2} , 10^{-3} and 10^{-4}); a) $\varepsilon = 0.2$, b) $\varepsilon = 1$, $M=10$.
- Fig. 9: Profiles of τ_{zz} along the axial mid-plane of the filament, fe -Lax, at a Hencky-strain of $\varepsilon = 1$, $\Delta t_{Hencky} = 10^{-2}$ and 10^{-4} ; accuracy in time for different ratios of Hencky-strain step Δt_{Hencky} to the inner time-step Δt_{inner} ($\Delta t_{Hencky} / \Delta t_{inner} = 10, 50, 100, 500$ and 1000); a) fe -Lax, b) fv -Lax.
- Fig. 10: Comparison of different fv -schemes, modified area-weighting ($\alpha_l^T - lhs$ version) and pure fluctuation distribution fv -Lax, transient development of τ_{zz} up to a Hencky-strain level of $\varepsilon = 1$, low-polymeric viscosity fluid, $\Lambda_0 = 2/3$; a) for node n_2 , b) for the free-surface mid-point node n_1 .
- Fig. 11: Profiles of τ_{zz} along the mid-plane section, comparison between fixed and dynamic Hencky-strain step, at a Hencky-strain level of unity; a) pure fluctuation distribution fv -Lax, (b) modified area-weighting ($\alpha_l^T - lhs$ version).
- Fig. 12: Field plots for velocity components, at $\varepsilon = 1$, $\Lambda_0 = 2/3$, low polymeric viscosity fluid, modified area-weighting scheme ($\alpha_l^T - lhs$ version), a) V_r , b) V_z , c) p , d) τ_{zz} .

- Fig. 13: Field plots for velocity components, at $\varepsilon = 0.2$, $\Lambda_0 = 2/3$, high-polymeric viscosity fluid, modified area-weighting scheme ($\alpha_l^T - lhs$ version), ($N=300$, $M=10$, $\text{tol}=10^{-6}$); a) V_r , b) V_z .
- Fig. 14: Field plots for velocity components, at $\varepsilon = 0.2$, $\Lambda_0 = 2/3$, high-polymeric viscosity fluid, modified area-weighting scheme ($\alpha_l^T - lhs$ version), ($N=1000$, $M=10$, $\text{tol}=10^{-6}$); a) V_r , b) V_z .
- Fig. 15: Profiles of τ_{zz} along the mid-plane axis of the filament, at a Hencky-strain of $\varepsilon = 0.2$, high-polymeric viscosity fluid, modified area-weighting scheme ($\alpha_l^T - lhs$ version).
- Fig. 16: Profiles of τ_{zz} along the mid-plane axis of the filament, at a Hencky-strain of $\varepsilon = 1$, low versus high-polymeric viscosity fluid, initial aspect-ratio of $\Lambda_0 = 2/3$, modified area-weighting scheme ($\alpha_l^T - lhs$ version).
- Fig. 17: Profiles of τ_{zz} along the centreline of the liquid bridge, at a Hencky-strain of $\varepsilon = 0.2$, low-polymeric viscosity fluid, $\Lambda_0 = 2/3$ and $1/3$, modified area-weighting scheme ($\alpha_l^T - lhs$ version).
- Fig. 18: Profiles of τ_{zz} along the mid-plane section of the filament, low-polymeric viscosity fluid, $\Lambda_0 = 2/3$ and $1/3$, modified area-weighting scheme ($\alpha_l^T - lhs$ version); a) $\varepsilon = 0.2$, b) $\varepsilon = 1$.
- Fig. 19: Field plots for $\Lambda_0 = 1/3$, modified area-weighting scheme, low polymeric viscosity fluid, at $\varepsilon = 1$; a) V_r , b) V_z , c) p , d) τ_{zz} .
- Fig. 20: Profile of the radial velocity V_r along the free-surface line at a Hencky-strain level of $\varepsilon = 1$. Comparison with Yao and McKinley [25], $\Lambda_0 = 1/3$.
- Fig. 21: Profile of the axial velocity V_z along the centreline of the liquid bridge at a Hencky-strain level of $\varepsilon = 1$. Comparison with Yao and McKinley [25], $\Lambda_0 = 1/3$.
- Fig. 22: Transient development of the axial normal force F_z at the moving end-plates. Comparison with Yao and McKinley [25] and Lubrication theory, up to a Hencky-strain level of $\varepsilon = 1$, $\Lambda_0 = 1/3$.
- Fig. 23: Development of the minimum radius $R_{\min}$ in Hencky-strain, comparison between initial aspect ratios of $\Lambda_0 = 1/3$ and $2/3$ and lubrication approximation ($\text{Exp}(-0.75\varepsilon)$).

	High-polymeric viscosity fluid	Low-polymeric viscosity fluid
ρ (density)	1030 (kg m ⁻³)	894 (kg m ⁻³)
L_0 (initial length)	12.7*10 ⁻³ (m)	12.7*10 ⁻³ (m)
Initial aspect ratio	2/3	2/3, 1/3
χ (surface-tension coefficient)	30.0*10 ⁻³ (Nm ⁻¹)	28.5*10 ⁻³
μ_2 (solvent viscosity)	0.069 (Pa.s)	43.2 (Pa.s)
μ_1 (polymer viscosity)	25.8 (Pa.s)	4.0 (Pa.s)
μ_0 (zero shear viscosity)	25.869 (Pa.s)	47.2 (Pa.s)
Relaxation time (λ_1)	0.421 s	2.54 s
Reynolds number	0.08	0.006
Weissenberg number (We)	5.08	5.08
Ca^{-1}	0.007	0.02

Table 1: Filament and fluid characteristics.

Fig. 1: (a) fe with 4 fv -sub-cells Ω_T , (b) mdc area for node l .

Fig. 2: Domain of filament, full length model with boundary conditions, and local disc movement definition.

a) τ_{zz} along the mid-plane axis of the filament ($z=0$), $\varepsilon = 0.2$

b) Transient development of τ_{zz} in time for the free-surface mid-point (n_1)

Fig. 3: Comparison between Lax and LDB schemes: a) τ_{zz} along the mid-plane section of the filament at $\varepsilon = 0.2$, b) transient development of τ_{zz} in time for the free-surface mid-point (node n_1), $\varepsilon = 0$ to 0.2.

Fig. 4: Profiles of total stress T_{zz} along the radial centreline of the liquid bridge, comparison with Yao and McKinley [25], at a Hencky-strain level of $\varepsilon = 0.2$, low-polymeric viscosity fluid.

Fig. 5: Comparison of different fv -schemes, fv -Lax, modified area-weighting ($\alpha_l^T - lhs$ version) and CT3, τ_{zz} along the mid-plane section of the filament at a Hencky-strain level of $\varepsilon = 1$, low-polymeric viscosity fluid.

Fig. 6: Transient development of τ_{zz} for the free-surface mid-point node (n_1), up to a Hencky-strain level of 0.2, comparison of *fe*-based schemes (Galerkin, SUPG and *fe*-Lax).

a) low-polymeric viscosity fluid

b) high-polymeric viscosity fluid

Fig. 7: Comparison between fv -Lax and fe -Lax schemes, at a Hencky-strain level of $\varepsilon = 0.2$; τ_{zz} along the mid-plane axis of the filament ($z=0$); a) low-polymeric viscosity fluid, b) high-polymeric viscosity fluid.

a) $\varepsilon = 0.2$

b) $\varepsilon = 1$.

Fig. 8: Profiles of τ_{zz} along the axial mid-plane axis of the filament, *fe*-Lax, accuracy in time for different Hencky-strain steps (10^{-2} , 10^{-3} and 10^{-4}); a) $\varepsilon = 0.2$, b) $\varepsilon = 1$, $M=10$.

a) *fe*-Lax

b) *fv*-Lax

Fig. 9: Profiles of τ_{zz} along the axial mid-plane of the filament, *fe*-Lax, at a Hencky-strain of $\varepsilon = 1$, $\Delta t_{\text{Hencky}} = 10^{-2}$ and 10^{-4} ; accuracy in time for different ratios of Hencky-strain step Δt_{Hencky} to the inner time-step Δt_{inner} ($\Delta t_{\text{Hencky}} / \Delta t_{\text{inner}} = 10, 50, 100, 500$ and 1000), a) *fe*-Lax, b) *fv*-Lax.

a) $\tau_{zz}(t)$ for node n_2

b) $\tau_{zz}(t)$ for the free-surface mid-point node n_1

Fig. 10: Comparison of different fv -schemes, modified area-weighting ($\alpha_l^T - lhs$ version) and pure fluctuation distribution fv -Lax, transient development of τ_{zz} up to a Hencky-strain level of $\varepsilon = 1$, low-polymeric viscosity fluid, $\Lambda_0 = 2/3$; a) for node n_2 , b) for the free-surface mid-point node n_1 .

a) pure fluctuation distribution (fv -Lax)

b) modified area-weighting scheme (α_l^T -lhs version)

Fig. 11: Profiles of τ_{zz} along the mid-plane section, comparison between fixed and dynamic Hencky-strain step, at a Hencky-strain level of unity; a) pure fluctuation distribution fv -Lax, (b) modified area-weighting (α_l^T -lhs version).

Fig. 12: Field plots for velocity components, at $\varepsilon = 1$, $\Lambda_0 = 2/3$, low polymeric viscosity fluid, modified area-weighting scheme (α_l^T - lhs version), a) V_r , b) V_z , c) p , d) τ_{zz} .

Fig. 13: Field plots for velocity components, at $\varepsilon = 0.2$, $\Lambda_0 = 2/3$, high-polymeric viscosity fluid, modified area-weighting scheme ($\alpha_l^T - lhs$ version), ($N=300$, $M=10$, $\text{tol}=10^{-6}$); a) V_r , b) V_z .

Fig. 14: Field plots for velocity components, at $\varepsilon = 0.2$, $\Lambda_0 = 2/3$, high-polymeric viscosity fluid, modified area-weighting scheme ($\alpha_l^T - lhs$ version), ($N=1000$, $M=10$, $\text{tol}=10^{-6}$); a) V_r , b) V_z .

Fig. 15: Profiles of τ_{zz} along the mid-plane axis of the filament, at a Hencky-strain of $\varepsilon = 0.2$, high-polymeric viscosity fluid, modified area-weighting scheme (α_l^T - *lhs* version).

Fig. 16: Profiles of τ_{zz} along the mid-plane axis of the filament, at a Hencky-strain of $\varepsilon = 1$, low versus high-polymeric viscosity fluid, initial aspect-ratio of $\Lambda_0 = 2/3$, modified area-weighting scheme (α_l^T - *lhs* version).

Fig. 17: Profiles of τ_{zz} along the centreline of the liquid bridge, at a Hencky-strain of $\varepsilon = 0.2$, low-polymeric viscosity fluid, $\Lambda_0 = 2/3$ and $1/3$, modified area-weighting scheme (α_l^T - *lhs* version).

a) $\epsilon = 0.2$

b) $\epsilon = 1$

Fig. 18: Profiles of τ_{zz} along the mid-plane section of the filament, low-polymeric viscosity fluid, $\Lambda_0 = 2/3$ and $1/3$, modified area-weighting scheme ($\alpha_l^T - lhs$ version); a) $\epsilon = 0.2$, b) $\epsilon = 1$.

Fig. 19: Field plots for $\Lambda_0 = 1/3$, modified area-weighting scheme, low polymeric viscosity fluid, at $\varepsilon = 1$; a) V_r , b) V_z , c) p , d) τ_{zz} .

Fig. 20: Profile of the radial velocity V_r along the free-surface line at a Hencky-strain level of $\varepsilon = 1$. Comparison with Yao and McKinley [25], $\Lambda_0 = 1/3$.

Fig. 21: Profile of the axial velocity V_z along the centreline of the liquid bridge at a Hencky-strain level of $\varepsilon = 1$. Comparison with Yao and McKinley [25], $\Lambda_0 = 1/3$.

Fig. 22: Transient development of the axial normal force F_z at the moving end-plates. Comparison with Yao and McKinley [25] and Lubrication theory, up to a Hencky-strain level of $\varepsilon = 1$, $\Lambda_0 = 1/3$.

Fig. 23: Development of the minimum radius R_{min} in Hencky-strain, comparison between initial aspect ratios of $\Lambda_0 = 1/3$ and $2/3$ and lubrication approximation ($\text{Exp}(-0.75\varepsilon)$).