STABILISATION TECHNIQUES FOR VISCOELASTIC FLOWS

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Abstract. In a comparative study, three different procedures of stabilisation techniques are considered for incompressible viscoelastic flow. The standard benchmark of flow past a sphere falling in a cylindrical vessel is employed to test these stabilisation techniques. First, discontinuity capturing (for stress) has been employed to investigate stabilisation properties. Then, the Zienkiewicz - Zhu (ZPR-Q) quadratic patch recovery (for velocity gradients) is analysed and compared with a direct recovery technique. Finally, a third procedure is based on strain-rate stabilisation (for momentum). Results demonstrate that discontinuity capturing itself yields reasonable stabilisation properties, and quadratic patch recovery improves the solution somewhat over the linear version, but is not as stable as direct averaging. Strain-rate stabilisation also provides enhanced stabilisation properties comparable to those of discontinuity capturing.

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1 INTRODUCTION

The stabilisation of numerical methods for viscoelastic flows is an active area of research, in the pursuit of improved stability and accuracy properties for the underlying solvers. Although there are concerns over stability due to incompressibility conditions (LBB-conditions), when using finite element techniques, overcoming such difficulties for viscoelastic problems is reasonably well understood. In contrast, stability conditions arising from solving the constitutive equations are not so well-established. Further research is necessary here, in order that more reliable numerical solution procedures may be developed, see Dupret [1] on loss of evolution.

There are many finite element solution techniques available to solve viscoelastic flow problems [2, 3, 4, 5, 6, 7]. Some of these use several thousand elements to solve small-scale problems, such as the settling sphere problem, in order to obtain an accurate solution. However, such an approach, without addressing stabilisation characteristics on coarse meshes, would be difficult to employ for large-scale computations. It is important to remark here, that solvers should be constructed in such a manner as to yield superior accuracy, even on relatively coarse meshes. Such a goal may only be realised by establishing the accuracy of various stabilisation techniques.

In this article, three different forms of stabilisation procedures are considered: namely, discontinuity capturing (DC), ZPR-Q-quadratic patch recovery and strain-rate (SRS) stabilisation. All three methods have been employed to solve the classical benchmark problem of settling flow, that of flow past a sphere falling in a cylinder of viscoelastic fluid. Each method is compared against its contenders, and also, with other available numerical solutions from the literature. All major quantities, such as stress, velocity gradient and pressure distribution around the sphere surface, are computed and compared.

In our earlier publications [6, 8], we have established that recovery procedures can yield stable solutions that reflect a high-degree of accuracy. Yet, further investigation into different methods is still necessary to derive an optimal implementation of such procedures to stabilise typical viscoelastic flow solvers. The first stabilisation approach considered is based on the application of discontinuity capturing (DC) techniques, applied to stress. The DC-method is intended to capture sharp features in the underlying solution and to suppress discretisation error in their corresponding representations. These solution features may take the form of discontinuities or sharp solution gradients [5]. Two DC- implementations are considered: a consistent and an inconsistent version. In the former approach, the finite element weighting function for the stress equations has an extra component. This acts in the direction of the solution gradient, and consequently, handles sharp gradients (layers) in the flow. In contrast, with the inconsistent implementation, an additional term is appended to the weighted residual equations. This is rationalised as follows. For the localised zones of interest, the most important section in the stress equations corresponds to the interaction of the DC-test function and the advection term itself. It is our experience that both DC-implementations yield a stable scheme, with
similar behaviour, and superior performance beyond that of conventional versions.

The second procedure employed in this study is to investigate a quadratic form of the Zienkiewicz and Zhu [9] patch-recovery method. In this method a quadratic-fit of the gradients of velocity is established over a pre-constructed patch of elements on each time-step. These accurate velocity gradients, recovered from patches, are embedded within the computation in such a manner as to extract, not only accurate, but also stable solutions. The results obtained demonstrate the improved stabilisation characteristics of the quadratic patch-recovery method beyond those pertaining to linear patch-recovery methods.

The third method considered in this study is that of strain-rate stabilisation for momentum. In this method, the difference between the continuous and discontinuous (discrete and accurate form) representations of the continuity equation is appended to the momentum equations (see Guénée and Fortin [10] under DEVSS schemes, Gong and Gijer [11] for iterative stability). The continuous component is calculated using a gradient recovery procedure [8]. The discontinuous form is directly taken from the finite element solution at each time-step. This scheme may be implemented in an adaptive manner. Then, local variations in the finite element size across the domain may be incorporated to retain the additional diffusive stabilisation component in a local manner. This avoids pollution of the solution in relatively smooth sections of the flow. The results reveal that this form of strain-rate stabilisation technique does indeed improve the stability of the solution procedure. In comparison to the quadratic-patch recovery technique, strain-rate stabilisation can yield more accurate and stable solutions, for example, reaching higher Deborah numbers of about 1.3 for the falling sphere problem.

We proceed to contrast the various methodologies outlined above, to identify the relative strengths and weaknesses of each scheme proposed.

2 Governing Equations

The non-dimensional form of the governing equations for incompressible viscoelastic flows can be written as:

**Conservation of mass**

\[ \nabla \cdot \mathbf{u} = 0. \]  \hspace{1cm} \text{(1)}

**Conservation of momentum**

\[ Re \frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot \tau - Re \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p, \]  \hspace{1cm} \text{(2)}

where \( \mathbf{u} \) is velocity, \( \tau \) extra stress, \( p \) the pressure, and \( t \) is the time variable. A Reynolds number, \( Re \), is defined as

\[ Re = \frac{\rho UL}{\mu}, \]  \hspace{1cm} \text{(3)}
where, \( \rho \) is the density of the fluid, \( U \) and \( L \) are characteristic velocity and length scales, respectively; \( \mu \) is the viscosity of the fluid. Fluid viscosity can be split into two parts: a polymeric contribution (\( \mu_1 \)) and a solvent (Newtonian) part (\( \mu_2 \)).

The non-dimensional constitutive equations of Oldroyd-B type can be expressed as

\[
De \frac{\partial \tau}{\partial t} = 2\mu_1 \mathbf{D} - \tau - De(\mathbf{u} \cdot \nabla \tau - \nabla \cdot \mathbf{u} - (\mathbf{\tau} \cdot \nabla \mathbf{u})^\top).
\]

(4)

In the above equation the elasticity number emerges through a Deborah number, \( De \), defined as

\[
De = \frac{\lambda V_s}{R_s}.
\]

(5)

Here, \( \lambda \) is a relaxation time, \( V_s \), \( R_s \) are selected scales of velocity and radius of the sphere, respectively. \( \mathbf{D} \) is the deformation-rate tensor defined as

\[
\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top).
\]

(6)

Stress is interpreted via a transformation of the form

\[
\mathbf{\tau}_1 = \mathbf{\tau} + 2\mu_2 \mathbf{D}.
\]

(7)

In this work, the above equation set is solved through a Taylor-Petrov-Galerkin formulation with Recovery (see Matallah et al. [6, 8]).

3 Stabilisation Procedures

3.1 Discontinuity Capturing (DC)

Discontinuity capturing is often used in compressible flow calculations to capture strong discontinuities such as arise in the vicinity of shocks. A similar principal can be applied in viscoelastic flow computations, to smooth the solution, where it changes abruptly.

For the DC method, an additional term is appended to the streamline upwinding Petrov-Galerkin (SUPG) weighting function. The aim of this term is to act in the direction of the solution gradient. This assists in the discretisation of sharp solution gradients, wherever such apply. In addition, stability is achieved through such schemes over conventional alternatives. The weighting function may represented generally in the form

\[
\psi_i = \phi_i + \alpha^h \mathbf{u} \cdot \nabla \phi_i + \beta^h \mathbf{u}_p \cdot \nabla \phi_i,
\]

(8)

where

\[
\mathbf{u}_p = (\mathbf{u} \cdot \nabla \mathbf{r}^h / |\nabla \mathbf{r}^h|^2) \nabla \mathbf{r}^h
\]

(9)
\( \tau^h \) is the finite element solution of stress, \( \mathbf{u} \) is the velocity vector, \( \phi_i \) is the Galerkin weighting function, \( \alpha^h \) is the SUPG parameter and \( \beta^h \) is the DC parameter [5]. This version is consistent, since the weighting function is applied to all terms in the constitutive equations. For simplicity, a second version of the DC method is proposed by Shakib [12], that alters the formulation as follows. Instead of modifying the SUPG weighting function through Eq. 8, an additional term is added to the constitutive equation. This term acts in the direction of the solution gradients, and hence contributes in localised zones of the flow, where gradients are sharp. Such an approach may be implemented in several forms [13]. Here, we present that proposed by Shakib [12], given by

\[
\int \beta^h \mathbf{\nabla}_L \phi_i \cdot \mathbf{\nabla}_L \tau^h \, d\Omega. \tag{10}
\]

In this notation, the local area coordinates \( L_i \) for triangular elements are employed. Following Shakib,

\[
\mathbf{\nabla}_L = \begin{bmatrix} \partial / \partial L_1, \partial / \partial L_2, \partial / \partial L_3 \end{bmatrix}^\dagger. \tag{11}
\]

The parameter \( \beta^h \) is derived for two and three dimensions (see Shakib [12] and Baaijens [14]), viz

\[
\beta^h = \frac{2 \| \mathbf{\nabla} \tau^h \|^2}{\| \mathbf{\nabla}_L \tau^h \|^2}, \tag{12}
\]

where

\[
\| \mathbf{\nabla} \tau^h \|^2 = \mathbf{\Omega} \cdot \mathbf{\alpha}^h \mathbf{\Omega}^h, \quad \text{and} \quad \| \mathbf{\nabla}_L \tau^h \|^2 = \mathbf{\nabla}_L \tau^h \cdot \mathbf{\nabla}_L \tau^h,
\]

\[
\mathbf{\alpha}^h = \frac{1}{\sqrt{g}} \quad \text{and} \quad g = \frac{\partial L_i}{\partial x_j} \frac{\partial L_k}{\partial x_k} v_j v_k.
\]

\( v_j \) and \( \mathbf{\Omega} \tau^h \) represent the velocity and the residual of the interpolated constitutive equation, respectively. Carew et al. [5] found it necessary to place some restrictions upon the \( \beta^h \) factor, to ensure solution consistency, so that the additional term vanishes or the solution is smooth when necessary. The definition bounds the factor \( \beta^h \) and forces it to be less than unity,

\[
\beta^h = \begin{cases} 2 s^2 \alpha^h / \sqrt{b} & \text{for } \sqrt{2} s \leq b, \\ b^2 \alpha^h / 2 s^2 & \text{for } \sqrt{2} s > b \end{cases}, \tag{13}
\]

with \( s = \mathbf{\Omega} \tau^h \), \( b = \| \mathbf{\nabla}_L \tau^h \| \).

The inconsistent DC method was employed by Carew et al. [5] for 4:1 plane contraction flows, showing greater stability than conventional schemes. In this paper, we consider both consistent and inconsistent versions of the DC method, and comparison is made between
both. We note, here, that the inconsistent DC version is referred to as DC1, and the consistent form as DC2.

3.2 Superconvergent Patch Recovery-Zienkiewicz-Zhu method (ZPR-Q)

We investigate a patch recovery technique introduced by Zienkiewicz and Zhu [9]. In this approach, all mid-side node sample values contribute in the patch to a least squares fit. The solution at the vertices, recovered from the patch function, is then of superior accuracy to the underlying finite element solution. Such improved values may then supplant the original values in providing an improved finite element interpolant. It is clear that better interpolation quality would normally lead to improved stabilised solutions. The basic methodology may be found in Ref. [9]. In Matallah et al. [8], a linear version of the ZPR was employed. In contrast, in this paper, a quadratic approach is implemented. The polynomial expansion \( P \), valid over an element patch surrounding a vertex node, has the form \( [P] = [1, x, y, x^2, xy, y^2] \), giving six unknown parameters to be evaluated. Then a continuous solution may be given for each patch by

\[
u^o = [P] \cdot \{a\} = a_1 + a_2 \cdot x + a_3 \cdot y + a_4 \cdot x^2 + a_5 \cdot xy + a_6 \cdot y^2. \quad (14)
\]

Once \( \{a\}^T \) has been determined, given the coordinates of the vertex, a nodal value at this vertex may be calculated through Eq. 14. A discrete \( L_2 \) superconvergent recovery procedure is employed here, following Matallah et al. [8]. We note, that both vertices and mid-side nodal values are recovered through the patch. A unique value is gathered for vertices, whilst multi-values are generated for mid-side nodes, due to different patches sharing the node in question. An averaged value is evaluated as proposed by Zienkiewicz and Zhu [9].

3.3 Strain-rate Stabilisation (SRS)

Strain-rate stabilisation is achieved via a quantity calculated from the difference between the continuous and discontinuous (discrete and accurate form) representations for solvent stress, i.e. approximation to the deformation rate tensor (\( \dot{D} - D \)). The continuous representation is established from one of the gradient recovery procedures and the discrete solution is available directly from the finite element approximation (see Guénette and Fortin [10], Gong and Güçeri [11]). Essentially, in this method the following term is appended to the momentum equation:

\[
2\beta(\dot{D} - D) \quad (15)
\]

where \( \dot{D} \) is the approximate finite element rate of deformation tensor solution, and \( \beta \) is an adjustable parameter. In this paper, from several selections, we have found that \( \beta = 0.05 \) is suitable for the mesh used. However, a locally adapted form, \( \beta_h \), may be established, based on an element measure \( h \), using the relation recommended by Sun et al. [2].
\[ \beta_h = \frac{h}{h_i} \left| \frac{\tau_{ij}}{V_{max}} \right|_{max}. \] (16)

A thorough analysis of this localised \( \beta_h \) implementation has yet to be undertaken.

4 Problem specification

A well-known benchmark problem represented by the falling sphere in a viscoelastic fluid is considered. An Oldroyd-B model is chosen to model the viscoelastic fluid with the aim to compare and analyse different numerical schemes described in this study. The sphere is falling with a velocity \( V_0 \) in a cylindrical tube of fluid. The problem is considered to be axisymmetric, taking the frame of reference moving with the sphere, implying movement of tube wall with \(-V_0\). A Reynolds number is adopted, that implies creeping flow conditions, and the Deborah number \( \text{De} \) is scaled by \( \lambda V_0 / R_s \), where \( R_s \) represents the sphere radius. Boundary conditions are taken as follows. Vanishing radial velocity and shear stress are assumed along the centreline, with no-slip velocity on the sphere surface. Fully-relaxed stress is imposed at the inflow upstream, of a plug flow nature. The cylinder is considered sufficiently long that fully-developed flow conditions are imposed on the downstream outflow. Upstream and downstream lengths are 13 \( R_s \) and 20 \( R_s \), respectively. A vanishing pressure datum is specified at the downstream exit. The ratio of the sphere radius to the cylinder radius is 0.5. Identical material parameters are selected to reflect the literature (see Lunsmann et al. [15] and Chauvière and Owens [16]), such as the ratio of the solvent viscosity \( \mu_s \) to the zero-shear viscosity \( \mu_0 \), \( \mu_s / \mu_0 = 0.5 \). A schematic flow diagram for the problem and two finite element meshes are displayed in Figure 1. Throughout this study, a time-step is chosen of \( \Delta t = 10^{-3} \), unless otherwise stated, and an \( l_2 \) relative temporal increment norm termination tolerance, \( \epsilon \), for velocity, pressure and stress is \( 10^{-6} \). The termination test is defined as

\[ \frac{\| \tau^{n+1} - \tau^n \|}{1 + \| \tau^{n+1} \|} \leq \epsilon, \] (17)

where superscripts \( n \) and \( (n + 1) \) represent successive time levels. Two meshes are considered, where mesh refinement is performed locally around the sphere surface. The number of elements around the sphere are 36 and 80 elements for mesh \( M_1 \) and \( M_2 \), respectively, both of uniform distribution.

We commence our simulations for the DC methods, analysing mesh convergence. Subsequently, mesh \( M_2 \) is employed thereafter.

5 Results and Discussion

Both versions of DC-method are considered in this analysis. In Figure 2, the \( \tau_{zz} \) stress component is compared between DC1 (inconsistent) and DC2 (consistent) schemes employing mesh \( M_1 \), with \( \text{De} = 0.9 \). Agreement is observed between the two schemes. This
gives confidence that the inconsistent approach (DC1) is a respectable discrete approximation, removing doubt concerning degradation of accuracy. Having established this point, only inconsistent discontinuity capturing (DC1) is considered subsequently, in our scope of results.

To establish accuracy for the inconsistent DC scheme, we employ two meshes $M_1$ and $M_2$, described earlier. As in Figure 3, for $De = 0.9$, oscillations are present in all four stress components on mesh $M_1$. These are removed on mesh $M_2$. This is apparent for $\tau_{rr}$ in the wake, and on the sphere surface for $\tau_{xz}$ and $\tau_{zz}$. On this basis, mesh $M_2$ is considered as quite adequate to represent accurate results. The core flow field does not reflect any visual differences between the solutions on the two meshes.

When discontinuity capturing DC1 is employed with the recovery scheme on mesh $M_2$, the maximum Deborah number of 1.3 is reached, as compared to 1.2 for the recovery scheme alone. For low $De$ values ($De \leq 0.6$), all stress components are smooth. Oscillations begin at values of $De \geq 0.9$, in $\tau_{zz}$ component on the upper part of the sphere quadrant ($-0.5 < z < 0.5$), as observed in Figure 4. At $De = 1.3$, the stress component $\tau_{zz}$ goes negative within the wake of the sphere (rear-stagnation point). Numerical divergence ensues. A line plot of $\tau_{zz}$ is plotted along the centreline axis, around the sphere and through the wake in Figure 4, that demonstrates this effect with increasing $De$. Such solution structure may be resolved with local mesh refinement around the wake of the sphere.

Figure 5 illustrates results obtained from the ZPR-Q method on Mesh $M_2$ for $De = 0.6$ and 1.2. At $De = 0.6$, stress components along the sphere surface are smooth. However, at $De = 1.2$, the results show oscillations and a rough distribution of stress is observed on the sphere surface. Although the same convergence criteria of the previous section was satisfied at $De = 1.2$, the ZPR-Q failed to produce a smooth solution. When we compare the DC1 and ZPR-Q schemes at a low level of elasticity ($De = 0.6$), the ZPR-Q method is smoother than that with the DC1 approach, as displayed for stress components in Figure 6 on the sphere surface. This implies that superior accuracy is achieved with the ZPR-Q scheme, whilst improved stability resides with the DC1 alternative.

Finally, when strain-rate stabilisation is employed, a Deborah number of $De = 1.2$ is reached. The scheme is convergent and stable with $\beta$ of 0.05. In contrast, above this Deborah number level, the solution failed to converge. Stress components are plotted in Figure 7 along the sphere surface and through the wake. Adjusting the parameter $\beta$ from 0.05 to 0.1 did not affect the solution. For $\beta = 0.5$, the scheme is oscillatory at $De = 1.2$ leading to inaccurate representation of the solution in the wake, as compared with the literature, see Figure 7. Stress components are also displayed for the adaptive strain-rate stabilisation method in Figure 8. There is no discernible difference between solutions with either adaptive strain-rate stabilisation or SRS with $\beta = 0.05$. This may be due to the mesh structure, producing adaptive values around 0.05 in the regions of significance. The adaptive approach permits the problem (with its mesh) to automatically select its level and variation of $\beta_k$. Clearly this is preferable to an empirical estimation.
Figure 9 contrasts the quantity of solutions generated at peak $De$ of 1.2 for both DC1 and SRS ($\beta = 0.05$) schemes. It is clear from the principal stress component $\tau_{zz}$, that the DC1 scheme provides the smoother solution profiles, and in this respect is the superior performance choice.

6 Conclusions

Different stabilisation techniques are analysed for the falling sphere problem. Both consistent and inconsistent DC methods were implemented and agreement is observed between both versions. Employing the inconsistent DC1 approach with recovery, enabled higher $De$ values of 1.3 to be reached, compared with 1.2 for the recovery scheme without discontinuity capturing. The scheme was stable and convergent at this level of $De$. Negative values in $\tau_{zz}$ are observed for $De = 1.3$ in the wake, that lead to a temporal periodic state beyond $De = 1.3$. Mesh refinement in the wake may resolve this particular barrier.

In contrast, the ZPR-Q scheme was convergent to a Deborah number limit of $De = 1.2$, but displayed oscillations in the stress components. Strain-rate stabilisation on the other hand, generates smooth stress profiles and the scheme is stable and convergent at $De = 1.2$. Of the procedures considered, DC is found superior in stabilisation performance, though adaptive strain-rate stabilisation may also compete with this, as they are closely related implementations. Overall, at low elasticity levels, the ZPR-Q scheme provides superior accuracy properties. In contrast, at high Deborah numbers, improved stability characteristics are captured with the DC1 method. Possibly, a combination of ZPR-Q and DC1 may blend this performance together, an avenue left for further exploration.
REFERENCES


**List of Figures**

1. Schematic flow diagram and finite element meshes for falling sphere problem. 12
2. Comparison of schemes DC1 and DC2, mesh $M_1$, $De = 0.9$. 12
3. Stress components on the sphere and the wake, mesh convergence $M_1$ and $M_2$, inconsistent DC (DC1), $De = 0.9$. 13
4. Stress components on the sphere and the wake, increasing $De$ for mesh $M_2$, inconsistent DC (DC1). 14
5. Stress components on the sphere and the wake, increasing $De$ for mesh $M_2$, ZPR-Q. 15
6. Stress components on the sphere surface, comparison of schemes DC1 and ZPR-Q, $De = 0.6$, mesh $M_2$. 16
7. Stress components on the sphere and the wake, mesh $M_2$ $De = 1.2$, Strain-rate stabilisation (SRS) for $\beta = 0.05$, 0.1, 0.5 and adaptive. 17
8. Stress components on the sphere surface, mesh $M_2$ $De = 1.2$, Strain-rate stabilisation (SRS) for $\beta = 0.05$ and adaptive. 18
9. Stress components on the sphere surface, comparison of schemes DC1 and Strain-rate stabilisation (SRS) ($\beta = 0.05$), $De = 1.2$, mesh $M_2$. 19
Figure 1: Schematic flow diagram and finite element meshes for falling sphere problem.

Figure 2: Comparison of schemes DC1 and DC2, mesh $M_1$, $De = 0.9$. 
Figure 3: Stress components on the sphere and the wake, mesh convergence $M_1$ and $M_2$, inconsistent DC (DC1), $De = 0.9$. 
Figure 4: Stress components on the sphere and the wake, increasing $De$ for mesh $M_2$, inconsistent DC (DC1).
Figure 5: Stress components on the sphere and the wake, increasing $De$ for mesh $M_2$, ZPR-Q.
Figure 6: Stress components on the sphere surface, comparison of schemes DC1 and ZPR-Q. 
$D_e = 0.6$, mesh $M_2$. 
Figure 7: Stress components on the sphere and the wake, mesh $M_2\ De = 1.2$, Strain-rate stabilisation (SRS) for $\beta = 0.05, 0.1, 0.5$ and adaptive.
Figure 8: Stress components on the sphere surface, mesh $M_2 \text{De} = 1.2$, Strain-rate stabilisation (SRS) for $\beta = 0.05$ and adaptive.
Figure 9: Stress components on the sphere surface, comparison of schemes DC1 and Strain-rate stabilisation (SRS) ($\beta = 0.05$). $De = 1.2$, mesh $M_2$. 

Axial distance, $z$