Discrete Temporal Constraint Satisfaction Problems*

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Introduction

A *constraint satisfaction problem* is a computational problem where the input consists of a finite set of variables and a finite set of constraints, and where the question is whether there exists a mapping from the variables to some fixed domain such that all the constraints are satisfied. When the domain is finite, and arbitrary constraints are permitted in the input, the CSP is NP-complete. However, when only constraints for a restricted set of relations are allowed in the input, it might be possible to solve the CSP in polynomial time. The set of relations that is allowed to formulate the constraints in the input is often called the *constraint language*. The question which constraint languages give rise to polynomial-time solvable CSPs has been the topic of intensive research over the past years. It has been conjectured by Feder and Vardi [6] that CSPs for constraint languages over finite domains have a complexity dichotomy: they are in P or NP-complete.

A famous CSP over an infinite domain is *feasibility of systems of linear inequalities over the integers*. It is of great importance in practice and theory of computing, and NP-complete. In order to obtain a systematic understanding of polynomial-time solvable restrictions and variations of this computational problem, Jonsson and Lööw [7] proposed to study the class of CSPs where the constraint language Γ is definable in *Presburger arithmetic*; that is, it consists of relations that have a first-order definition over $(\mathbb{Z}; \leq, +)$. The constraint satisfaction problem for Γ, denoted by $\mathrm{CSP}(\Gamma)$, is the problem of deciding whether a given conjunction of formulas of the form $R(y_1, \ldots, y_n)$, for some n-ary R from Γ, is satisfiable in Γ. By appropriately choosing such a constraint language Γ, a great variety of problems over the integers can be formulated as $\mathrm{CSP}(\Gamma)$. An example of a CSP in this class is $\mathrm{CSP}(\mathbb{Z}; +, 1)$, which is the problem of deciding whether a system of linear diophantine equations has a solution (this problem is solvable in polynomial time [5]).

In the present work, we study the constraint satisfaction problems whose constraint language is first-order definable over $(\mathbb{Z}; <)$. These problems are called *discrete temporal CSPs*. The class of discrete temporal CSPs properly

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contains the class of *temporal CSPs* (whose constraint language is first-order definable over $(\mathbb{Q}; <)$ [1]) and of *distance CSPs* (whose constraint language if definable over $(\mathbb{Z}; \mathrm{succ})$, where succ is the binary successor relation [2]). Our main result shows that the class of discrete temporal CSPs exhibits a P/NP-complete dichotomy (modulo the Feder-Vardi conjecture for finite-domain CSPs; several authors claimed recently to have proved this conjecture [8, 4, 9]).

A cornerstone of our proof is the characterization of those problems that are discrete temporal CSPs but that are neither temporal CSPs nor finite-domain CSPs; the corresponding constraint languages have an interesting notion of *rank* which we use to obtain a strong pre-classification of those languages up to homomorphic equivalence. The notion of rank is central to reduce the classification to the natural special case where the binary successor relation or its symmetric closure is part of the language:

Theorem 1. *Let Γ be a finite-signature structure that is definable over $(\mathbb{Z}; <)$. Then there exists a structure Δ with the same CSP as Γ, and such that one of the following cases applies:*

1. *Δ is a finite structure,*
2. *Δ is definable over $(\mathbb{Q}; <)$,*
3. *Δ is definable over $(\mathbb{Z}; <)$ and can simulate the relation $|y - x| = 1$,*
4. *Δ is definable over $(\mathbb{Z}; <)$ and can simulate the relation $y - x = 1$.*

We remain vague on the precise meaning of "simulate". The important property of this simulation is that adding the simulated relation to Δ does not change the complexity of the CSP, so that we can focus on those templates Δ that actually contain one of the given relations.

The Universal-Algebraic Approach

The proof of this theorem relies on the so-called universal-algebraic approach; this is the first time that this approach has been used for constraint languages that are not finite or countably infinite *ω-categorical*. The central insight of the universal-algebraic approach to constraint satisfaction is that the computational complexity of a CSP is captured by the set of *polymorphisms* of the constraint language. One of the ideas of the present article is that in order to use polymorphisms when the constraint language is not ω-categorical, we have to pass to the countable saturated model of the first-order theory of $(\mathbb{Z}; <)$. We briefly review these notions here, as well as the central statement (Theorem 2) explaining the importance of countable saturated structures for the universal-algebraic approach.

A *polymorphism* of a relational structure Γ is a homomorphism from some finite direct power of Γ to Γ. An *endomorphism* of Γ is a homomorphism $\Gamma \to \Gamma$, and an *automorphism* is a bijective endomorphism whose inverse is also an endomorphism.

A countable structure Γ is *saturated* if for every infinite set of formulas $\Phi(\overline{x})$ in finitely many free variables, if every finite subset $\Phi'(\overline{x})$ of $\Phi(\overline{x})$ is satisfiable in Γ, then $\Phi(\overline{x})$ itself is satisfiable in Γ. The structure $(\mathbb{Z};<)$ is not saturated: the set $\Phi(x,y) := \{\exists z_1,\dots,z_n(x < z_1 < z_2 < \cdots < z_n < y) \mid n \geq 1\}$ is finitely satisfiable in $\mathbb{Z}$ but is not globally satisfiable. However this structure has a natural saturated extension $(\mathbb{Q}\times\mathbb{Z};<)$, which consists of the lexicographic ordering on $\mathbb{Q}\times\mathbb{Z}$. Additionally, every structure that is definable over $(\mathbb{Z};<)$ has a corresponding saturated extension that has the same CSP and that is definable over $(\mathbb{Q}\times\mathbb{Z};<)$.

Theorem 2. *Let Δ be definable in a countable saturated structure Γ. Let R be a relation that has a first-order definition in Γ. Then:*

- *R is preserved by all the automorphisms of Δ iff R is first-order definable in Δ,*
- *R is preserved by all the endomorphisms of Δ iff R is existentially positively definable in Δ,*
- *If R is the union of n orbits of tuples under the automorphisms of Γ, then R is preserved by all the n-ary polymorphisms of Δ iff R is primitively positively definable in Δ.*

The Classification

For discrete temporal CSPs, the border between polynomial-time tractability and NP-completeness can be described using the following operations. A *square isomorphism* is an isomorphism from $(\mathbb{Q}\times\mathbb{Z};\mathrm{succ})^2$ to $(\mathbb{Q}\times\mathbb{Z};\mathrm{succ})$. For $d \geq 1$, the *d-modular maximum* is the binary operation

$$(x,y) \mapsto \begin{cases} \max(x,y) & \text{if } x = y \mod d \\ x & \text{otherwise.} \end{cases}$$

The *d-modular minimum* is defined similarly.

Theorem 3. *Let Γ be a finite-signature structure that is definable over $(\mathbb{Z};<)$. Assume that Γ contains the relation $y = x+1$ or the relation $|y-x| = 1$. Then at least one of the following cases apply:*

1. *The countably saturated extension of Γ admits a (equivalently, all) square isomorphism as a polymorphism, and* $\mathrm{CSP}(\Gamma)$ *is in P,*
2. *For some $d \geq 1$, Γ admits the d-modular maximum or the d-modular minimum as a polymorphism, and* $\mathrm{CSP}(\Gamma)$ *is in P, or*
3. $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

Our classification has a particularly simple form when the constraint language Γ not only contains the binary successor relation, but also the relation

$x < y$: if Γ has the polymorphism $(x, y) \mapsto \max(x, y)$ or $(x, y) \mapsto \min(x, y)$, then CSP($\Gamma$) is in P, and is NP-complete otherwise.

Putting together Theorems 1 and 3, as well as the results from [1], we obtain a full complexity classification for the constraint satisfaction problems of structures definables over $(\mathbb{Z}; <)$, modulo the Feder-Vardi conjecture.

References

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