Filtration versus Team Semantics

Martin Lück[1]

1 **Leibniz Universität Hannover, Institut für Theoretische Informatik, Appelstraße 4, 30167 Hannover, Germany**
lueck@thi.uni-hannover.de

Team semantics, introduced by Hodges [8], enjoys striking success as a compositional semantics for logics with incomplete information, like independence-friendly logic by Hintikka and Sandu [7]. Compositionality is achieved by letting each interpretation encode a multitude of possible states simultaneously. Such a macro-state is then called a *team*.

In this vein, Väänänen [13] introduced *dependence logic* as an extension of first-order logic. It additionally provides logical atoms that relate the states in a team to each other. It allows statements of the form

"The value of x is a function of y, but does not depend on z."

Here we use the symbolic notation $T \vDash {=}(y, x)$ ("y determines x") and $T \nvDash {=}(z, x)$ ("z does not determine x"), where T, a team, is a set of assignments to the variables in question. Clearly, for such a statement to make sense, the evaluation must extend over a multitude of instances of x, y and z, and indeed the dependence atom ${=}(\cdot)$ has no counterpart in usual first-order logic.

A rich family of logics of dependence and independence has been developed, and dependence logic and its variants have found a broad range of applications like database theory, quantum mechanics and statistics. Having an inherent second-order flavor due to their semantics, their complexity is however vastly higher than their classical counterparts. The validity problem for first-order sentences using dependence atoms is, for instance, non-arithmetical, while it is recursively enumerable in the absence of such atoms [13].

The concept of team semantics has been introduced into other logics as well, like modal logic [12], propositional logic [14] and quantified Boolean logic [4]. These logics, which all are decidable, have been intensively studied in terms of computational complexity, expressive power, and possible proof systems.

We study *modal team logic ($\mathcal{MTL}$)*, which itself is far from being completely understood. Its model checking problem is classified as PSPACE-complete by Müller [11], and a finite model property is established by Kontinen et al. [9]. However, the computational complexity of its satisfiability problem is still unknown. Moreover, no upper or lower bounds for minimal model size are established. In contrast, the classical modal logic is well-understood: it has the *exponential model property*, i.e., every formula of length n that is satisfiable has a model of size at most 2^n. This bound is optimal under a suitable encoding of modal formulas [6].

We pursue a model-theoretic approach to team semantics via the successful *filtration* technique. Filtration has turned out to be a powerful tool to prove the exponential model property for a wide range of logics. Examples are modal logics [2], dynamic and temporal logics [1], and even fragments of first-order logic [3].

We employ the standard definition of a Kripke structure as a triple (W, R, V), where (W, R) is a directed graph and V maps to each propositional symbol p a subset $V(p)$ of W.

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When working with *teams*, the semantics of $\mathcal{ML}$ is extended as follows:

$(\mathcal{K}, T) \vDash p$ iff $T \subseteq V(p)$
$(\mathcal{K}, T) \vDash \neg p$ iff $T \cap V(p) = \emptyset$
$(\mathcal{K}, T) \vDash \varphi \wedge \psi$ iff $(\mathcal{K}, T) \vDash \varphi$ and $(\mathcal{K}, T) \vDash \psi$
$(\mathcal{K}, T) \vDash \varphi \vee \psi$ iff $\exists S, U \subseteq T$ such that $T = S \cup U$, $(\mathcal{K}, S) \vDash \varphi$, and $(\mathcal{K}, U) \vDash \psi$
$(\mathcal{K}, T) \vDash \Diamond\varphi$ iff $\exists S \subseteq W$ such that $(\mathcal{K}, S) \vDash \varphi$,
$\forall u \in T\, \exists v \in S : (u, v) \in R$, and $\forall v \in S\, \exists u \in T : (u, v) \in R$
$(\mathcal{K}, T) \vDash \Box\varphi$ iff $(\mathcal{K}, S) \vDash \varphi$, where $S := \{ v \mid (u, v) \in R, u \in T \}$

The above definitions of the connectives admit the so-called *flatness property*, which states that team semantics and the classical *point semantics* coincide in the following sense.

▶ **Proposition 1** (Flatness). *If $\varphi \in ML$, then $(\mathcal{K}, T) \vDash \varphi$ if and only if $\forall w \in T : (\mathcal{K}, w) \vDash \varphi$.*

The full modal team logic $\mathcal{MTL}$ is obtained by adding the Boolean negation $\sim$ to the syntax, and defining $(\mathcal{K}, T) \vDash {\sim}\varphi$ if and only if $(\mathcal{K}, T) \nvDash \varphi$.

The basic idea of filtration is that worlds in a Kripke structure, using the standard Kripke semantics, are only distinguishable by $\mathcal{ML}$ formulas up to a certain degree. Usually, this is modeled as an equivalence relation $\approx$ on W. A natural such relation is *bisimulation*, as two bisimilar worlds u, v cannot be distinguished by any model formula, i.e., $(\mathcal{K}, u) \vDash \varphi \Leftrightarrow (\mathcal{K}, v) \vDash \varphi$ for arbitrary $\mathcal{ML}$ formulas φ.

Bisimulation up to bounded modal depth yields finite, but still huge models. Instead, the following relation is used, which can be understood as a courser approximation of bisimulation.

▶ **Definition 2.** Let $\mathcal{K} = (W, R, V)$ be a Kripke structure and $\Gamma \subseteq \mathcal{ML}$ finite. For $u, v \in W$, define $u \approx_\Gamma v$ if and only if u and v agree on the formulas in Γ, i.e., $\forall \varphi \in \Gamma : (\mathcal{K}, u) \vDash \varphi \Leftrightarrow (\mathcal{K}, v) \vDash \varphi$.

Indeed $\approx$ is an equivalence relation on W, and furthermore it has index at most $2^{|\Gamma|}$. This is also the cardinality of $W/_{\approx_\Gamma} := \{ [w]_{\approx_\Gamma} \mid w \in W \}$, the set of equivalence classes of worlds in W.

To define a whole Kripke structure with worlds $W/_{\approx_\Gamma}$, we require a set R' of edges and a propositional valuation V'. A proposition p is simply assumed true in $[w]_{\approx_\Gamma}$ if it is true in at least one w' with $w \approx_\Gamma w'$. Likewise, we choose the edges R' as follows. Whenever an edge $(u, v) \in R$ exists, then we add $([u]_{\approx_\Gamma}, [v]_{\approx_\Gamma})$ to R'. This definition is referred to as the *minimal filtration of* R, and the whole structure $(W/_{\approx_\Gamma}, R', V')$ is then written $\mathcal{K}/_{\approx_\Gamma}$, and called *the filtration of* $\mathcal{K}$ *through* Γ.

The crucial property of filtration is the following.

▶ **Theorem 3** (Filtration Lemma, [2]). *Let $\Gamma \subseteq \mathcal{ML}$ be closed under taking subformulas. Let $\mathcal{K} = (W, R, V)$ be a Kripke structure. Then, for all formulas $\varphi \in \Gamma$ and all worlds $w \in W$, it holds $(\mathcal{K}, w) \vDash \varphi$ if and only if $(\mathcal{K}/_{\approx_\Gamma}, [w]_{\approx_\Gamma}) \vDash \varphi$.*

The following question immediately arises for team semantics:

Given $\varphi \in \mathcal{MTL}$, and any model $(\mathcal{K}, T)$ of φ, does there exist an equivalence relation $\approx$ with sufficiently small index such that $(\mathcal{K}_\approx, [T]_\approx) \vDash \varphi$?

We are able to answer the question positively for certain fragments of $\mathcal{MTL}$, but negatively for the full logic.

First of all, a suitable extension of $[\cdot]$ (we sometimes drop the index from now on) from worlds to teams is required. A straightforward definition is $[T] := \{ [w] \mid w \in T \}$. Indeed, it is easily shown from the filtration lemma that $\approx_\Gamma$ works for $\mathcal{ML}$ formulas also in team semantics.

▶ **Definition 4.** If $\mathcal{K} = (W, R, V)$ is a Kripke structure, $\varphi \in \mathcal{MTL}$, and $\approx$ is an equivalence relation on W, then $\approx$ is called *φ-invariant in $\mathcal{K}$* if

$$(\mathcal{K}, T) \vDash \varphi \Leftrightarrow (\mathcal{K}/_\approx, [T]_\approx) \vDash \varphi$$

for all teams $T \subseteq W$. Moreover, $\approx$ is *strongly φ-invariant in $\mathcal{K}$* if every refinement of $\approx$ is φ-invariant in $\mathcal{K}$.

▶ **Proposition 5.** *For every $\varphi \in \mathcal{ML}$ and every Kripke structure, there exists a strongly φ-invariant equivalence relation with index at most $2^{|\varphi|}$.*

For formulas without the flatness property, the matter is more complicated. We are however able to show by induction that invariance is inherited along Boolean connectives and also the "splitting disjunction" $\vee$, where the induction hypothesis has to be applied to the subteams forming a division of T.

▶ **Lemma 6.** *If $\approx$ is strongly φ-invariant and strongly ψ-invariant on a Kripke structure $\mathcal{K}$, then it is strongly ξ-invariant on $\mathcal{K}$ for $\xi \in \{\sim\varphi, \varphi \wedge \psi, \varphi \vee \psi\}$.*

It follows that two fragments of $\mathcal{MTL}$, called $\mathcal{B}(\mathcal{ML})$ and $\mathcal{S}(\mathcal{ML})$, have the exponential model property. $\mathcal{B}(\mathcal{ML})$ is the *Boolean closure* of $\mathcal{ML}$, $\mathcal{ML}$ closed under application $\wedge$ and $\sim$, while $\mathcal{S}(\mathcal{ML})$ is defined as the closure of $\mathcal{ML}$ under $\wedge, \sim$ and $\vee$.

Unfortunately, the introduction of team-wide modalities in non-flat formulas immediately destroys this brief success.

▶ **Theorem 7.** *For every n there is an $\mathcal{S}(\mathcal{ML})$ formula φ of length $\mathcal{O}(n)$, and a Kripke structure $\mathcal{K}$, such that every equivalence relation on $\mathcal{K}$ that is strongly invariant for $\Box\varphi$ or $\Diamond\varphi$ has at least index 2^{2^n}.*

Proof idea. We prove only the $\Box$ case, as the proof still goes through for $\Diamond$. We employ the formula $\max(n)$, which is true in a team T only if all possible 2^n Boolean assignments to the propositions $x_1, \ldots, x_n$ occur in worlds of T [5]. Let $\mathcal{K}$ contain these 2^n worlds, and call this subset of worlds S. Then $(\mathcal{K}, S) \vDash \max(n)$. Add 2^{2^n} further worlds to $\mathcal{K}$, one for each possible configuration to have a subset of S as successors. Call this subset T. Now suppose an equivalence relation $\approx$ does not distinguish between two worlds $x, y \in T$. W.l.o.g. x has some successor $s \in S$ which y does not have. Consequently, the set T' of all worlds in T not having s as successor, including y, does still not satisfy $\Box\max(n)$. $[T]$, however, has the successor s, due to the merging of x and y, and therefore satisfies $\Box\max(n)$. ◀

This theorem shows that filtration, despite being successful for standard modal logic, does not easily translate to team semantics: intuitively, there are too many possibilities for a world to make a non-satisfying team T satisfying when being merged into it, or vice versa. The formulas $\max(n)$, $\Box\max(n)$ and $\Diamond\max(n)$ all possess models of exponential size, but these cannot be reached by filtration.

To establish the exponential model property for a logic, the filtration method is actually unnecessarily strong. It is possible to give up the symmetric definition of φ-invariance, which is actually the preservation of φ and $\sim\varphi$ in all teams. Moreover, given a model $(\mathcal{K}, T)$, it is sufficient to preserve φ in T instead of *all* teams of $\mathcal{K}$.

Consequently, we weaken the definition of filtration as follows:

▶ **Definition 8.** If $(\mathcal{K}, T)$ is a model of $\varphi \in \mathcal{MTL}$, and $\approx$ is an equivalence relation on the worlds of $\mathcal{K}$, then $\approx$ is called *φ-preserving in* $(\mathcal{K}, T)$ if $(\mathcal{K}/_{\approx}, [T]_{\approx}) \vDash \varphi$. Moreover, $\approx$ is *strongly φ-preserving in* $(\mathcal{K}, T)$ if every refinement of $\approx$ is φ-preserving in $(\mathcal{K}, T)$.

It is possible to apply this definition to a larger fragment than $\mathcal{S}(\mathcal{ML})$ (which has strongly invariant filtration). We call this fragment $\mathcal{MTL}_{\mathsf{mon}}$, and it is defined as the closure of $\mathcal{S}(\mathcal{ML})$ under all operators except $\sim$ (with additionally Boolean disjunction $\varphi \otimes \psi := \sim(\sim\varphi \wedge \sim\psi)$ allowed).

▶ **Theorem 9.** *Let $\varphi \in \mathcal{MTL}_{mon}$. For every model $(\mathcal{K}, T)$ of φ there exists an equivalence relation $\approx$ with index at most $2^{|\varphi|}$ such that $(\mathcal{K}/_{\approx}, [T]_{\approx}) \vDash \varphi$.*

Since $\mathcal{MTL}_{\mathsf{mon}}$ allows modalities only in the form of existential quantification, it is not very surprising that it admits exponential models. Consequently, it turns out that the addition of universal quantifications, by allowing $\Diamond$ to occur negatively, does not even permit the above weak notion of filtration.

▶ **Theorem 10.** *For every n there is an $\mathcal{MTL}$ formula φ of length $\mathcal{O}(n)$, and a model $(\mathcal{K}, T)$ of φ, such that every equivalence relation on the worlds of $\mathcal{K}$ that is strongly φ-preserving has at least index $n!^{(n)}$.*

Here, $n!^{(n)} := n\underbrace{!\ldots!}_{n \text{ times}}$ poses a non-elementary lower bound for filtrations.

To conclude, these results show that the filtration technique utterly fails as an adequate tool for full $\mathcal{MTL}$. Of course it is still unknown if $\mathcal{MTL}$ has the exponential model property, and all formulas used in the theorems have, but such small models cannot be reached by the means of simply forming the filtration of arbitrary existing models.

Nevertheless, we achieved progress on grasping the complexity of modal team logic: $\mathcal{MTL}$ is much more succinct than the fragments considered here, despite being expressively equivalent to $\mathcal{B}(\mathcal{ML})$ [10].

▶ **Corollary 11.** *Every $\mathcal{MTL}$ formula has an equivalent $\mathcal{B}(\mathcal{ML})$ formula, but for every n there in an $\mathcal{MTL}$ formula φ of length $\mathcal{O}(n)$ such that every equivalent $\mathcal{B}(\mathcal{ML})$ formula has length at least $n!^{(n)}$.*

It is noteworthy that the satisfiability problem of any fragment of $\mathcal{MTL}$ with an exponential model property is in AEXPTIME(poly), the class of problems decidable in exponential runtime with polynomially many alternations. Nevertheless, the translation of $\mathcal{MTL}$ to the already AEXPTIME(poly)-complete fragment $\mathcal{S}(\mathcal{ML})$ is non-elementary. This indicates that the complexity of $\mathcal{MTL}$ is much higher, otherwise implying that $\mathcal{MTL}$ is vastly more succinct but unable to actually encode meaningful information.

Further research aims at solving this question and settling the computational complexity of the satisfiability problem of $\mathcal{MTL}$.

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