Slicewise definability in first-order logic with bounded quantifier rank *

Yijia Chen
School of Computer Science
Fudan University
yijiachen@fudan.edu.cn

Jörg Flum
Mathematisches Institut
Universität Freiburg
joerg.flum@math.uni-freiburg.de

Xuangui Huang
Department of Computer Science
Shanghai Jiao Tong University
stslxg@gmail.com

Let φ be a sentence of first-order logic FO. The *quantifier rank* of φ, denoted by $\text{qr}(\varphi)$, is the maximum nested depth of quantifiers in φ. If φ defines a graph property K, that is,

$$\text{K} = \big\{\mathcal{G} \bigm| \mathcal{G} \text{ a graph and } \mathcal{G} \text{ has the property } \varphi\big\},$$

then a straightforward algorithm can decide whether an input graph $\mathcal{G}$ belongs to K in time $O(|\mathcal{G}|^{\text{qr}(\varphi)})$. Therefore, minimizing the quantifier rank of φ would lead to better algorithms for deciding the graph property K. Many graph properties amount to the existence of a certain set of vertices of size k, where k is a fixed constant. A well-known example is the *k-vertex-cover problem* of deciding whether a given graph $\mathcal{G}$ contains a set C of k vertices such that every edge in $\mathcal{G}$ has at least one end in C. The set C is then called a *k-vertex-cover* of $\mathcal{G}$. Clearly, the existence of a k-vertex-cover can be expressed by the following sentence of FO

$$\psi_k := \exists x_1 \cdots \exists x_k \left(\bigwedge_{1\le i<j\le k} x_i \neq x_j \wedge \forall u \forall v \big(Euv \to \bigvee_{i=1}^{k} (u = x_i \vee v = x_i)\big)\right).$$

In other words, a graph $\mathcal{G}$ has a k-vertex-cover if and only if $\mathcal{G}$ satisfies ψ_k. Observe that $\text{qr}(\psi_k) = k+2$, hence the naive algorithm derived from ψ_k would have running time $O(|\mathcal{G}|^{k+2})$. Clearly it is far worse than the existing linear time algorithms for the k-vertex-cover problem. An immediate question is whether the k-vertex-cover problem can be defined by a sentence φ_k with $\text{qr}(\varphi_k) < k+2$. As the first main result of this paper we show that this is in indeed possible for a φ_k with $\text{qr}(\varphi_k) \le 17$. Note that this holds for every k even though we need different φ_k's for different k's. The k-vertex-cover problem is the *kth slice* of the parameterized vertex cover problem

p-VERTEX-COVER	
Input:	A graph $\mathcal{G}$.
Parameter:	k.
Question:	Does $\mathcal{G}$ have a vertex cover of size k?

*The full version of the paper is available at `https://arxiv.org/abs/1704.03167`.

For $q \in \mathbb{N}$ we denote by FO_q the class of FO-sentences of quantifier rank at most q. Our result can be phrased in terms of the *slicewise definability* [7] of p-VERTEX-COVER:

Theorem 1. *p-VERTEX-COVER is slicewise definable in FO_{17}.*

The vertex cover problem is a special case of the *hitting set problem* on hypergraphs of bounded hyperedge size. For every $d \in \mathbb{N}$ a d-hypergraph is a hypergraph with hyperedges of size at most d. Then, the *parameterized d-hitting set problem* p-d-HITTING-SET asks whether an input d-hypergraph $\mathcal{G}$ contains a set of k vertices that intersects with every hyperedge in $\mathcal{G}$. Thus p-VERTEX-COVER is basically the parameterized 2-hitting set problem. Extending Theorem 1 we prove

Theorem 2. *Let $d \geq 1$. Then p-d-HITTING-SET is slicewise definable in FO with bounded quantifier rank; more precisely, p-d-HITTING-SET is slicewise definable in FO_q with $q = O(d^2)$.*

Let $\varphi(X)$ be an $\mathrm{FO}[\tau]$-formula which for a, say r-ary, second-order variable X may contain atomic formulas of the form $Xx_1 \dots x_r$. Then the *parameterized problem* $\mathrm{FD}_{\varphi(X)}$ *Fagin-defined by* $\varphi(X)$ is the problem

$\mathrm{FD}_{\varphi(X)}$	
Input:	A τ-structure $\mathcal{A}$.
Parameter:	$k \in \mathbb{N}$.
Question:	Decide whether there is an $S \subseteq A^r$ with $\lvert S \rvert = k$ and $\mathcal{A} \models \varphi(S)$.

The problem p-d-HITTING-SET can be Fagin-defined [6] by an appropriate FO-formula. Thereby we view a hypergraph $\mathcal{G} := (V, E)$ as an $\{E_0, \varepsilon\}$-structure $\big(V \cup E, E, \varepsilon^{\mathcal{G}}\big)$, where E_0 is a unary relation symbol and ε is a binary relation symbol and

$$E_0^{\mathcal{G}} := E \quad \text{and} \quad \varepsilon^{\mathcal{G}} := \big\{(v, e) \;\big|\; v \in V, e \in E \text{ and } v \in e\big\}.$$

Fix $d \in \mathbb{N}$. For $k \in \mathbb{N}$ we have (assuming $\lvert V \rvert \geq k$)

$$(\mathcal{G}, k) \in p\text{-}d\text{-HITTING-SET} \quad \iff \quad \text{for some } S \text{ with } \lvert S \rvert = k \text{ we have } (V \cup E, E, \varepsilon^{\mathcal{G}}) \models \varphi(S),$$

where

$$\varphi(X) := \forall e \Big(E_0 e \to \forall x_1 \dots \forall x_d \big((\forall x (x \,\varepsilon\, e \leftrightarrow \bigvee_{i=1}^{d} x_i = x) \to (Xx_1 \vee \dots \vee Xx_d) \big) \Big).$$

Observer that in the above $\varphi(X)$ the second-order variable X does not occur in the scope of an existential quantifier or negation symbol. We show that all problems Fagin-definable in this form are slicewise definable in some FO_q, which improves [8, Theorem 4.4].

Theorem 3. *Let $\varphi(X)$ be an $\mathrm{FO}[\tau]$-formula without first-order variables occurring free and in which X does not occur in the scope of an existential quantifier or negation symbol. Then $\mathrm{FD}_{\varphi(X)} \in \mathrm{XFO}_{\mathrm{qr}}$ (where, by definition, $\mathrm{XFO}_{\mathrm{qr}}$ is the class of problems slicewise definable in FO_q for some $q \in \mathbb{N}$).*

What is the complexity of the class of parameterized problems that are slicewise definable in FO with bounded quantifier rank? We prove that it coincides with para-FO [4], the class of problems FO-definable after a precomputation on the parameter.

Theorem 4. para-FO $=$ $\mathrm{XFO}_{\mathrm{qr}}$.

Thus we obtain a descriptive characterization of the class para-FO, or equivalently of the parameterized circuit complexity class para-AC^0 [5, 2, 4].

The equivalence between para-FO and para-AC^0 is an easy consequence of the equivalence between FO and the classical circuit complexity class uniform AC^0 [3]. This equivalence crucially relies on the assumption that the input graphs (or more generally, the input structures) are equipped with built-in addition and multiplication. In fact, the main technical tool for proving Theorem 1 and the subsequent results, the *color-coding* method [1], makes essential use of arithmetic. Without addition and multiplication, it is not difficult to show that p-VERTEX-COVER cannot be slicewise defined in FO_q for any $q \in \mathbb{N}$. Thus Theorem 1 exhibits the power of addition and multiplication, although on the face of it, the vertex cover problem has nothing to do with arithmetic operations.

In finite model theory there is consensus that inexpressibility results for FO and for fragments of FO are very hard to obtain in the presence of addition and multiplication. To get such a result we exploit the equivalence between FO and uniform AC^0, more precisely, we analyze the connection between the quantifier rank of a sentence φ and the depth of the corresponding AC^0 circuits. Together with a theorem [9, 10] on a version of Sipser functions we show that the hierarchy $(\mathrm{FO}_q)_{q\in\mathbb{N}}$ is strict:

Theorem 5. *Let $q \in \mathbb{N}$. Then there is a parameterized problem slicewise definable in* FO_{q+1} *but not in* FO_q.

References

[1] N. Alon, R. Yuster, and U. Zwick. Color-coding. *Journal of the ACM*, 42(4):844–856, 1995.

[2] M. Bannach, C. Stockhusen, and T. Tantau. Fast parallel fixed-parameter algorithms via color coding. In *Proceedings of the 10th International Symposium on Parameterized and Exact Computation, IPEC 2015, September 16-18, 2015, Patras, Greece*, pages 224–235, 2015.

[3] D. A. Mix Barrington, N. Immerman, and H. Straubing. On uniformity within NC^1. *Journal of Computer and System Sciences*, 41(3):274–306, 1990.

[4] Y. Chen and J. Flum. Some lower bounds in parameterized AC^0. In *Proceedings of the 41st International Symposium on Mathematical Foundations of Computer Science, MFCS 2016, August 22-26, 2016 - Kraków, Poland*, pages 27:1–27:14, 2016.

[5] M. Elberfeld, C. Stockhusen, and T. Tantau. On the space and circuit complexity of parameterized problems: Classes and completeness. *Algorithmica*, 71(3):661–701, 2015.

[6] J. Flum and M. Grohe. Fixed-parameter tractability, definability, and model-checking. *SIAM Journal on Computing*, 31(1):113–145, 2001.

[7] J. Flum and M. Grohe. Describing parameterized complexity classes. *Information and Computation*, 187(2):291–319, 2003.

[8] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006.

[9] J. Håstad. Almost optimal lower bounds for small depth circuits. In *Randomness and Computation*, pages 6–20. JAI Press, 1989.

[10] N. Segerlind, S. R. Buss, and R. Impagliazzo. A switching lemma for small restrictions and lower bounds for k-DNF resolution. *SIAM Journal on Computing*, 33(5):1171–1200, 2004.