Focusing light logics

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***Abstract*—This talk deals with an ongoing work on focusing proofs of light logics. After reviewing focalization in linear logic and a particular proof method, we turn to the question of focalization for light logics, focusing on ELL in this abstract. We provide a focusing proof system for ELL and sketch the focalization theorem. Finally, we plan to discuss questions that we are currently investigating as well as directions for future work.**

I. Introduction

a) Parallel vs. sequential proof objects: The rise of linear logic put forward two approaches to linear proofs, which aim at more canonical proofs, usually referred to as the parallel approach and the sequential approach. The parallel approach corresponds to proof nets while the sequential approach draw from the study of Andreoli's focusing theorem. Both approach can be viewed as reducing the amount of inessential non-determinism of sequent proofs: proof nets quotient part of the non-determinism of cut-elimination, namely commutative steps of cut-elimination, while focused proofs quotient part of the non-determinism of proof-search, by committing to a focus in the positive phase.

While proof-nets appeared as objects of the first generation (they have a substantial part in Girard's seminal paper [6]), focusing proofs can be viewed as a second-generation concept, appearing in Andreoli's thesis [1], [2] even though one can argue that focalization was partly anticipated in Miller et al's uniform proofs [11], [10] and goal-directed provability.

b) Light logics: Logic for bounded complexity drew increasing attention as variants of linear logic [8], [7], [5], [9] proved to be valuable tools for the intrinsic analysis of the computational complexity of programs. Their complexity properties are usually expressed with respect to proof-nets cut-elimination, that is with respect to the proof system made free from the commutative steps of cut-elimination.

Still, those logics are not so well understood from the point of view of structural proof theory. In this talk, we contribute to this direction by studying focusing in light logic.

c) Focusing in linear logic: Focusing has been recognized, since its introduction by Andreoli about 25 years ago, as a crucial structural property of (sequent) proof systems both for proof search (when aiming at automated theorem proving) and for the study of normal and canonical forms in sequent systems (as focused proofs are connected with pattern-matching).

Completeness proofs of focusing are usually complex, tedious and require to check many intermediate (and most often uninteresting) lemmas. Miller and Saurin [12] proposed a modular proof of focusing for LL which was later used for proving several other completeness results (for instance for logic with least and greatest fixed points [3]). Additionally, this proof, relying on the notion of focalization graph, was the starting point of the idea of multi-focusing which allows to focus on an arbitrary number of positive formulas and to recover a notion of canonicity thanks to maximally multifocused proofs [4].

d) Organization of the abstract: We make use of this methodology, recalled in section II together with background on focusing, in order to establish a focusing result for ELL, sketched in section III before discussing in Section IV some ongoing work, open questions, as well as future work that we plan to discuss during the workshop.

II. Focusing proofs of linear logic

Alternating phases: To make the long story short, focalization expresses that, when searching for proofs, one can alternate between two strategies, reversibility and focusing, driven by the polarities of the formulas (or resources) in the sequent to be justified:

- Reversibility tells us that, when the sequent contains negative formulas, one can simply apply non-deterministically any negative rules.
- Focusing is concerned with how to justify a sequent containing no negative formulas: in such a situation, choose some positive formula (non atomic if such a formula exists), let us call it the *focus* and stick to this choice hereditarily, that is as long as the new sequents contain positive subformulas of the focus.

Non-deterministic search: Both phases of the search are non-deterministic: one needs to choose which negative formulas to activate in the reversible phase, which focus to choose and then which inference rules to apply in the focusing phase. Still, while negative non-determinism is unessential (and may be resolved by imposing an arbitrarily order on the selection of negative formulas), positive non-determinism is crucial. Indeed, while any step of the reversibility phase preserves provability (provability of the conclusion sequents ensures provability of the premises), the focusing phase forces us to commit to some crucial choices (such as which disjunct of an $\oplus$ to select) resulting in a potential loss of provability: a backtracking phase is therefore most likely to happen when searching for the focusing phase.

The real concern of focusing is not to escape this unavoidable backtracking but to dramatically restrict its magnitude by limiting the number of choice points in a focusing phase: a positive formula creates a choice point only when entering the focusing phase, while only positive inferences create choice points during the focusing phase.

Completeness of focused proofs: A focalization theorem therefore states that the subset of focused proofs is complete with respect to provability. More precisely, proved by means of permutability arguments, those focusing theorems ensure that to each LL proof, there is a denotationally equivalent focused proof (that is, a proof living in the same class of equivalence up to cut-elimination).

Many completeness proofs can be found in the literature, for LL or derived proof systems. We will recall, in the remaining of this section, the main ingredients of a proof given in a joint work with Miller [12]. The peculiarity of this proof is to identify the main elements of a proof of a focalization theorem in such a way as to identify some lemmas that, if satisfied, ensure focalization.

A modular proof of focalization: The following explanation sketches the main steps and will be developed in the talk. We refer the reader to [12] for more details.

1) Identify positive and negative connectives. Negative connectives/inference should permute downwards with any other (introduction rule of a) connective. Positive connectives should permute downwards with themselves. Those two kinds of permutability are respectively called strong and weak permutation. Strong permutation of the negatives ensures reversibility; the remaining of the proof deals with the focusing phase, and therefore with proofs concluded with a positive sequent.
2) Identify the positive trunk, which is essentially the maximal subtree of positive inferences. A positive trunk is therefore made only of positive inferences of the (active) formulae of the conclusion sequent (we omit the case of the cut rule in this description, it can be treated similarly). Considering this positive trunk (and the positive formulae active in this trunk) will be sufficient to find a candidate formula to stand as a focus, which is the role of the remaining steps.
3) Consider the sequents constituting the leaves of the positive trunk: they are all subformulas of active formulas of the conclusion and will be used to define the Focalization Graph of the proof.
4) Define the focalization graph as follows: its vertices are the formulae of the conclusion sequents which are active in the positive trunk, its edges connect A to B when, there is a sequent in the border containing a negative subformula of A and a positive subformula of B.
5) Focalization follows from the acyclicity of the focalization: those graphs therefore have a source which may be chosen as focus. Indeed, in any sequent of the border in which one of there subformula occurs, they occur negatively, that is there outermost positive cluster of connectives has been totally decomposed during the positive trunk and the result follows from weak permuatibility of the positive.
6) Acyclicity of the focalization graph follows from a series of lemmas which essentially amount to ensuring that (i) in any sequent of the positive trunk, there is at most one subformula of a given formula and (ii) in any branching rule, only one formula of the conclusion sequent has subformulas in both premises.

III. ELLF: Focusing Elementary Linear Logic

A. *A remark on the exponential rules*

Exponential rules are well-known to be both the most intriguing and richest part of linear logic proof theory and its least satisfying fragment[1]. As modalities, they are not canonical connectives, but they also contain most of the computational complexity of linear logic proofs since they control the use of weakening and contraction:

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\; ! \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\; ?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\; ?c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\; ?d$$

When it comes to focalization, the important fact is that they can be viewed as bipolar connectives and the fact that they delimit focusing phases. A standard approach is to work with dyadic sequents, that are sequent containing usual linear formulas as well as formulas implicitly prefixed with a $?$. Though not the only option, we follow this path in the present abstract.

B. *Various light logics.*

Among the three main proposals for light linear logics, LLL (Light linear logic [7]), ELL (Elementary linear logic [5]) and SLL (Soft linear logic [9]) we will focus[2], in the rest of this abstract on ELL. These three systems are obtained by modifying the exponential fragment of LL, multiplicative and additive fragments are unchanged (unless one desires to consider affine, additive-free, or intuitionistic fragments for instance, as is often the case in the literature).

The basic idea is that, in addition to usual rules for exponentials recalled above, another option is to replace promotion rule with the following pair of inferences functorial promotion and digging:

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\; !fonct \qquad \frac{\vdash \Gamma, ??A}{\vdash \Gamma, ?A}\; !dig$$

LL proof systems presented in this way in not convenient when searching for proofs and it would be contrived to formulate a focalization result in this setting. Luckily, the difficulty comes from digging and it will fade away in the following.

The complexity of cut-elimination in sequent calculi comes from the structural rules of weakening and contraction which are available in a controlled way in linear logic, thanks to the exponentials. It is no surprise, then, that light logics arise from restriction from the rules of linear logic, usually restricting the presentation of the exponential inferences with functorial promotion and digging; as is the case for ELL.

[1] Girard refers to them as the *imperfect world* playing on the double meaning of the word, in a linguistic and moral setting.

[2] No pun intended.

C. ELL: Elementary linear logic.

ELL Exponential rules are obtained by replacing promotion with functorial promotion and by omitting digging and dereliction:

$$\frac{\vdash \Gamma, A}{\vdash ?\,\Gamma, !\,A}\ !\,fonct \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?\,A}\ ?\,w \qquad \frac{\vdash \Gamma, ?\,A, ?\,A}{\vdash \Gamma, ?\,A}\ ?\,c$$

Full ELL sequent calculus is given in Figure 1. In this system, one cannot derive dereliction nor digging.

The fundamental result about ELL is that proof-net cut-elimination (for EAL, that is ELL with free weakening) has elementary complexity, resulting in the fact that EAL representable functions are elementary recursive functions.

We shall now explain how the method recalled in the previous section lead to focusing for ELL.

D. ELLF: Focusing ELL.

As usual, in order to treat the exponentials in a focused way, one moves to a dyadic system in order to treat weakening implicitly and to postpone contraction as much as possible by incorporating in multiplicative branching rules (*ie.* tensor and cut). This step actually decomposes inference rules for $?$ into a negative part and a positive part respectively. The resulting dyadic system is depicted in figure 2, where sequents have the form $\vdash \Sigma \mid \Gamma$, where all formulas of Σ are prefixed with a $?$.

One notices easily that in the dyadic system, the following permutatibility properties hold:

- $[\bot], [⅋], [\&], [\top], [\forall], [?]$ are negative (they have full permutability)
- $[\mathbf{1}], [\otimes], [\oplus_i], [\exists], [!]$ have weak permutability, and will therefore be considered as positive.

Positive and negative sequents are defined as usual.

Reversibility of negative rules is not problematic, let us concentrate on the focusing phase for positive sequents.

ELL positive trunks and focalization graphs: They are defined as follows:

Definition 1 (ELL-positive trunks)**.** *Let $\mathcal{S} = \vdash \Sigma \mid \Gamma$ be a positive sequent and $\mathfrak{D}$ a proof of this sequent. The positive trunk $\mathfrak{D}^+$ associated with $\mathfrak{D}$ is the maximal sub-tree of $\mathfrak{D}$ rooted in $\mathcal{S}$ such that:*

- *$\mathfrak{D}^+$ contains only positive inferences;*
- *If a sequent appearing in $\mathfrak{D}^+$ is conclusion of a functorial promotion rule, then it is a leaf of $\mathfrak{D}^+$.*

Definition 2 (ELL-Focalization Graphs)**.** *The focalization graph of an ELL proof of a positive sequent $\vdash \Sigma \mid \Gamma$ is defined as follows:*

- *Edges of the graph are formulas occurring in Γ which are active formulas in $\mathfrak{D}^+$;*
- *There is an edge from A to B if there exists a sequent, $\mathcal{S}$, of $\mathfrak{D}^+$ border containing a negative sub-occurrence of A and a positive sub-occurrence of B.*

The following proposition holds:

Proposition 3. *ELL-focalization graphs are acyclic.*

One can combine the previous results in a focusing theorem for ELL, which is expressed with respect to ELLF, proof system with explicit focusing annotations given in Figure 3:

Theorem 4 (ELL Focalization theorem)**.** *ELLF is complete for ELL provability. Moreover, for every ELL proof π, there is a focused proof (obtained by erasing the focusing annotations from an ELLF proof) equivalent to π.*

The sequents of ELLF contain explicit focusing annotations. They can be of two types: $\vdash \Psi\colon \Delta \Uparrow L$ and $\vdash \Psi\colon \Delta \Downarrow F$ representing respectively a sequent in the reversibility and the focusing phase, with the usual conditions on formulas occurring in Δ and the fact the L is a list of formulas.

IV. Discussion

The present abstract presented an ongoing work on focusing in light logic. We focused on ELL by adapting a proof technique from [12]. The study of focusing in light logic is of interest in at least two regards: first in better understanding the proof theory of light logics, second in helping us to better understand the nature of exponential modalities which are the tricky part of focusing, due to the fact that they are both partly positive and partly negative and that they delimit focusing phase.

Being an ongoing work, it leaves at least as many questions as it provides answers. There are a number of topics we plan to discuss during the workshop but had no space to develop here, or that we plan to address in the future, from the question of focusing for other light logics SLL and LLL, which raise different issues, to the possible applications to focused proof-search and logic programming with light exponential (or more probably mixed-exponentials), as well as other focusing proofs for light logics, typically by means of cut-elimination. Other natural questions would be whether there is an analogous of maximally multi-focused proofs in the case of ELL, but also, more broadly, whether the technique of [12] can be applied to other, non-light, variant of linear logic.

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$$\frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\ cut \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ [\bot] \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ [\parr] \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ [\otimes] \quad \frac{}{\vdash \mathbf{1}}\ [\mathbf{1}]$$

$$\frac{}{\vdash A, A^\perp}\ ini \qquad \frac{}{\vdash \Gamma, \top}\ [\top] \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ [\&] \qquad \frac{\vdash \Gamma, A_1}{\vdash \Gamma, A_1 \oplus A_2}\ [\oplus_0] \quad \frac{\vdash \Gamma, A_2}{\vdash \Gamma, A_1 \oplus A_2}\ [\oplus_1]$$

$$\frac{\vdash \Gamma, B}{\vdash ?\,\Gamma, !\,B}\ [!] \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?\,B}\ [?\,w] \qquad \frac{\vdash \Gamma, ?\,B, ?\,B}{\vdash \Gamma, ?\,B}\ [?\,c]$$

Figure 1: ELL sequent calculus

$$\frac{\vdash \Sigma \mid \Gamma, A \quad \vdash \Sigma \mid \Delta, A^\perp}{\vdash \Sigma \mid \Gamma, \Delta}\ cut \qquad \frac{\vdash \Sigma \mid \Gamma}{\vdash \Sigma \mid \Gamma, \bot}\ [\bot] \qquad \frac{\vdash \Sigma \mid \Gamma, A, B}{\vdash \Sigma \mid \Gamma, A \parr B}\ [\parr] \qquad \frac{\vdash \Sigma \mid \Gamma, A \quad \vdash \Sigma \mid \Delta, B}{\vdash \Sigma \mid \Gamma, \Delta, A \otimes B}\ [\otimes] \qquad \frac{}{\vdash \Sigma \mid \mathbf{1}}\ [\mathbf{1}]$$

$$\frac{}{\vdash \Sigma \mid A, A^\perp}\ ini \qquad \frac{}{\vdash \Sigma \mid \Gamma, \top}\ [\top] \qquad \frac{\vdash \Sigma \mid \Gamma, A \quad \vdash \Sigma \mid \Gamma, B}{\vdash \Sigma \mid \Gamma, A \,\&\, B}\ [\&] \qquad \frac{\vdash \Sigma \mid \Gamma, A_1}{\vdash \Sigma \mid \Gamma, A_1 \oplus A_2}\ [\oplus_0] \qquad \frac{\vdash \Sigma \mid \Gamma, A_2}{\vdash \Sigma \mid \Gamma, A_1 \oplus A_2}\ [\oplus_1]$$

$$\frac{\vdash \Sigma, ?\,B \mid \Gamma}{\vdash \Sigma \mid\ ?\,B, \Gamma}\ [?] \qquad \frac{\vdash \emptyset \mid F_1^{i_1}, \ldots, F_k^{i_k}, B}{\vdash ?\,F_1, \ldots, ?\,F_k \mid\ !\,B}\ [!] \quad \text{avec } i_1, \ldots, i_k \geq 0.$$

Figure 2: ELL Dyadic Sequent Calculus

$$\frac{\vdash \Psi\colon \Delta \Uparrow L}{\vdash \Psi\colon \Delta \Uparrow \bot, L}\ [\bot] \qquad \frac{\vdash \Psi\colon \Delta \Uparrow F, G, L}{\vdash \Psi\colon \Delta \Uparrow F \parr G, L}\ [\parr] \qquad \frac{}{\vdash \Psi\colon \Delta \Uparrow \top, L}\ [\top] \qquad \frac{\vdash \Psi\colon \Delta \Uparrow F, L \quad \vdash \Psi\colon \Delta \Uparrow G, L}{\vdash \Psi\colon \Delta \Uparrow F \,\&\, G, L}\ [\&]$$

$$\frac{}{\vdash \Psi\colon \cdot \Downarrow \mathbf{1}}\ [\mathbf{1}] \qquad \frac{\vdash \Psi\colon \Delta_1 \Downarrow F \quad \vdash \Psi\colon \Delta_2 \Downarrow G}{\vdash \Psi\colon \Delta_1, \Delta_2 \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Psi\colon \Delta \Downarrow F_i}{\vdash \Psi\colon \Delta \Downarrow F_1 \oplus F_2}\ [\oplus_i]$$

$$\frac{\vdash \Psi\colon \Delta \Uparrow F}{\vdash \Psi\colon \Delta \Downarrow F}\ R\Downarrow \qquad \frac{\vdash \Psi\colon \Delta, F \Uparrow L}{\vdash \Psi\colon \Delta \Uparrow F, L}\ R\Uparrow \qquad \frac{}{\vdash \Psi\colon X^\perp \Downarrow X}\ I \qquad \frac{\vdash \Psi\colon \Delta \Downarrow F}{\vdash \Psi\colon \Delta, F \Uparrow \cdot}\ D$$

$$\boxed{\frac{\vdash \Psi, F\colon \Delta \Uparrow L}{\vdash \Psi\colon \Delta \Uparrow ?\,F, L}\ [?]} \qquad \boxed{\frac{\vdash \cdot\colon \cdot \Uparrow \Psi', F}{\vdash \Psi\colon \cdot \Downarrow !\,F}\ [!] \quad \begin{array}{ll} \text{where} & \Psi = F_1, \ldots, F_k \\ \text{and} & \Psi' = F_1^{i_1}, \ldots, F_k^{i_k} \\ \text{with} & i_1, \ldots, i_k \geq 0 \end{array}}$$

Figure 3: ELLF: Focused system for ELL. *Constraints on inference rules are the following: In $R\Uparrow$, F is not negative while in $R\Downarrow$, F is either negative or a negative literal. In I, K is a positive literal. In D, F is not a negative literal.*

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